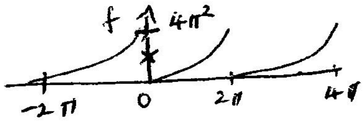
2H01 – Sheet 1 Answers to Q1 & Q2

1) Periodic function has graph



, Period 2π

If $x^2 = a_0 + \sum a_n \cos nx + b_n \sin nx$ (Period 2π), then, integrating $\int_0^{2\pi} x^2 dx = \int_0^{2\pi} a_0 dx + \ldots$

$\Rightarrow 2\pi a_0 = \frac{1}{3}.8\pi^3 \Rightarrow \underline{a_0 = 4\pi^2/3}$. Multiplying by $\cos mx$ & $\int_0^{2\pi}$ gives

$\int_0^{2\pi} x^2 \cos mx\, dx = 0 + a_m \int_0^{2\pi} \cos^2 mx\, dx = a_m . 2\pi . \frac{1}{2} \Rightarrow a_m = \frac{1}{\pi}\int_0^{2\pi} x^2 \cos mx\, dx.$

Similarly $b_m = \frac{1}{\pi}\int_0^{2\pi} x^2 \sin mx$. These can both be found using integration by parts, or you could do them together: $\pi(a_n + ib_n) = \int_0^{2\pi} x^2 e^{inx} dx = \left[x^2 \frac{e^{inx}}{in}\right]_0^{2\pi} -$

$\int_0^{2\pi} \frac{1}{in}.2x e^{inx} dx = \frac{4\pi^2.1}{in} + \left[\frac{2x}{n^2}e^{inx}\right]_0^{2\pi} - 2\cancel{\int_0^{2\pi} \frac{e^{inx}}{n^2}dx}^{\,0} = \frac{4\pi^2}{in} + \frac{4\pi}{n^2}$

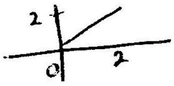
$e^{2\pi n i} = 1$

$\Rightarrow a_n + ib_n = \frac{4}{n^2} - \frac{4\pi i}{n} \Rightarrow a_n = 4/n^2,\ b_n = -4\pi/n$. As required.

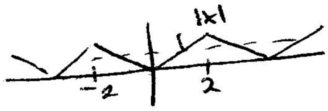
Fourier's Theorem guaranteed that at a discontinuity, such as at $x=0$, the series converges to the average $\frac{1}{2}(f(0+) + f(0-)) = \frac{1}{2}.(4\pi^2 + 0) = 2\pi^2$.

So $2\pi^2 = \frac{4\pi^2}{3} + \sum_{n=1}^{\infty}\left(\frac{4}{n^2}\cos n.0 - \frac{4\pi}{n}\sin n.0\right) = 4\sum_{n=1}^{\infty}\frac{1}{n^2} \Rightarrow \underline{\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}}$

2)



, Extend x to be an even function of x ($|x|$) & make periodic



. The period is 4 & the function is even so we expect the coefficients of $\sin x$ to be zero.

If $|x| = a_0 + \sum_1^{\infty} a_n \cos\frac{n\pi x}{2}$ ← Period 4, then $\int_{-2}^{2}|x|dx = 2\int_0^2 x\,dx = 2.\frac{2^2}{2} = 4 =$

$\int_{-2}^{2} a_0 dx + 0 = 4a_0 \Rightarrow a_0 = 1$.

Also $\int_{-2}^{2}|x|\cos\frac{m\pi x}{2}dx = 2\int_0^2 x\cos\frac{m\pi x}{2}dx = 0 + a_m\int_{-2}^{2}\cos^2\frac{m\pi x}{2}dx = a_m.\frac{1}{2}.4$

$\Rightarrow a_m = \int_0^2 x\cos\frac{m\pi x}{2} = \left[x\sin\left(\frac{m\pi x}{2}\right).\frac{2}{m\pi}\right]_0^2 - \frac{2}{m\pi}\int_0^2 \sin\left(\frac{m\pi x}{2}\right)dx = 2\overset{=0}{\sin(m\pi)}.\frac{2}{m\pi} +$

$\frac{4}{m^2\pi^2}\left[\cos\frac{m\pi x}{2}\right]_0^2 = \frac{4}{m^2\pi^2}(\cos m\pi - 1) = \frac{4}{m^2\pi^2}((-1)^m - 1) = \begin{cases} -\frac{8}{m^2\pi^2} & m \text{ odd} \\ 0 & m \text{ even}\end{cases}$

So $|x| = 1 - \frac{8}{\pi^2}\sum_{m \text{ odd}} \frac{\cos\left(\frac{m\pi x}{2}\right)}{m^2}$ & for $0<x<2$

$x = 1 - \frac{8}{\pi^2}\sum_{n=0}^{\infty}\frac{\cos((2n+1)\pi x/2)}{(2n+1)^2}$

...sevals theorem is, working it out from notes with $L=2$ rather than π, ~~on~~ ...,

$$\frac{1}{L}\int_{-L}^{L} f(x)^2 dx = 2a_0^2 + \sum_{1}^{\infty} a_n^2 + b_n^2.$$

& here this gives

$$\frac{1}{2}\int_{-2}^{2} |x|^2 dx = 2a_0^2 + \sum_{n=0}^{\infty} a_n^2 + b_n^2 .$$

$b_n = 0$, $a_0 = 1$, $L = 2$
$a_n = \frac{-8}{n^2\pi^2}$ n ~~even~~ odd.

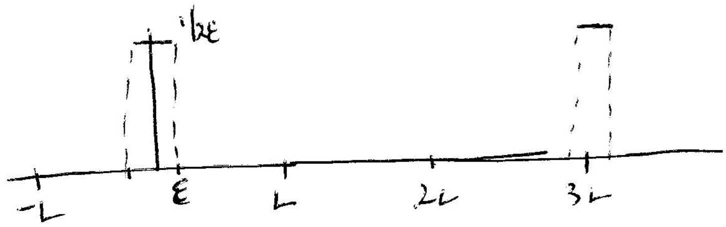
$$\Rightarrow \frac{1}{2}\int_{-2}^{2} x^2 dx = \frac{1}{3}.8 = 2.1^2 + \sum_{n\,odd} \frac{64}{\pi^4}\frac{1}{n^4} \Rightarrow \sum_{n\,odd} \frac{1}{n^4} = \frac{\pi^4}{64}\left(\frac{8}{3}-2\right)$$
$$= \frac{\pi^4}{96}$$

However the required sum is $S = \sum_{all\,n} \frac{1}{n^4} = \sum_{n\,odd} \frac{1}{n^4} + \sum_{n\,even} \frac{1}{n^4}$

The second sum is $\frac{1}{2^4} + \frac{1}{4^4} + \frac{1}{6^4} + \dots = \frac{1}{2^4}\left(1 + \frac{1}{2^4} + \frac{1}{3^4} + \dots\right) = \frac{1}{2^4}.S$

So $S\left(1 - \frac{1}{2^4}\right) = \frac{\pi^4}{96}$, $S = \frac{16}{15}.\frac{\pi^4}{96} = \underline{\frac{\pi^4}{90}.}$

Q5



- Even so coefficients of $\sin(n\pi x/L)$ are zero

If $f(x) = a_0 + \sum_1^\infty a_n \cos(n\pi x/L)$, then integrating

$$\int_{-L}^{L} f(x)\,dx = \int_{-L}^{L} a_0 + \sum_1^\infty a_n \cos\left(\frac{n\pi x}{L}\right) dx = 2La_0 + 0$$

$$\Rightarrow a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\,dx = \frac{1}{2L}\int_{-\varepsilon}^{\varepsilon} \frac{1}{2\varepsilon}\,dx = \frac{1}{2L}$$

Multiplying by $\cos\left(\frac{m\pi x}{L}\right)$ & integrating

$$\int_{-L}^{L} f(x)\cos\left(\frac{m\pi x}{L}\right) dx = 0 + \sum_1^\infty a_n \int_{-L}^{L} \cos\left(\frac{n\pi x}{L}\right)\cos\left(\frac{m\pi x}{L}\right) dx = a_m \cdot 2L \cdot \frac{1}{2}$$

$$\Rightarrow a_m = \frac{1}{L}\int_{-L}^{L} \underset{\uparrow \text{ even}}{f(x)\cos\left(\frac{m\pi x}{L}\right)} dx = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{m\pi x}{L}\right) dx = \frac{2}{L}\cdot\frac{1}{2\varepsilon}\int_0^\varepsilon \cos\left(\frac{m\pi x}{L}\right) dx$$

$$= \frac{\cancel{2}}{\cancel{L}}\cdot\frac{1}{\cancel{2}\varepsilon}\cdot\frac{\cancel{L}}{m\pi}\sin\left(\frac{m\pi\varepsilon}{L}\right)$$

$$f(x) = \frac{1}{2L} + \sum_1^\infty \frac{1}{n\pi\varepsilon}\sin\left(\frac{n\pi\varepsilon}{L}\right)\cos\left(\frac{n\pi x}{L}\right)$$

As $\varepsilon \to 0$, as $\frac{\sin\theta}{\theta} \to 1$ as $\theta \to 0$ we have

$$f(x) = \frac{1}{2L} + \frac{1}{L}\sum_1^\infty \cos\left(\frac{n\pi x}{L}\right)$$ which is <u>not</u> convergent

The Fourier coefficients do not $\to 0$ as $n \to \infty$. This is to be expected as the series is trying to represent a function which varies over arbitrarily small lengthscales ε.

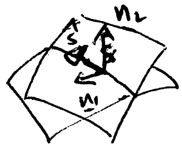
MATH2401 - Sheet 2 - Solutions

1 a) $z(x,y) = x^3+y^3-3xy$. $\nabla z = \begin{pmatrix}3x^2-3y\\3y^2-3x\end{pmatrix} = \begin{pmatrix}9\\-3\end{pmatrix}$ at $(1,1)$. Direction is $\underline{s} = \begin{pmatrix}2\\3\end{pmatrix}$ so $\hat{\underline{s}} = \frac{1}{\sqrt{13}}\begin{pmatrix}2\\3\end{pmatrix}$ & $\frac{\partial z}{\partial s} = \frac{1}{\sqrt{13}}\begin{pmatrix}2\\3\end{pmatrix}\cdot\begin{pmatrix}9\\-3\end{pmatrix} = 9/\sqrt{13}$

b) Direction has $y/x = \tan\pi/3 \Rightarrow \hat{s} = \begin{pmatrix}\cos\pi/3\\ \sin\pi/3\end{pmatrix} = \begin{pmatrix}1/2\\ \sqrt3/2\end{pmatrix}$. $z(x,y) = (y+1)^2 + x\cos(xy)$

$\nabla z = \begin{pmatrix}\cos(xy) - xy\sin(xy)\\ 2(y+1) - x^2\sin xy\end{pmatrix} = \begin{pmatrix}1\\2\end{pmatrix}$ at $(0,0) \Rightarrow \frac{\partial z}{\partial x} = \begin{pmatrix}1/2\\ \sqrt3/2\end{pmatrix}\cdot\begin{pmatrix}1\\2\end{pmatrix} = \frac12+\sqrt3$

c) $w = x^3+y^3+3z$, $\nabla w = 3x^2\underline{i} + 2y\underline{j} + 3\underline{k} = (3\underline{i} + 6\underline{j} + 3\underline{k})$ at $(1,3,2)$



$\underline{s} = \underline{n}_1\wedge\underline{n}_2$ with $\underline{n}_1$ & $\underline{n}_2$ the normals to the surfaces which can be found as gradients. $\underline{n}_1 = \nabla(x^2+y^2-2xz-6) = (2x-2z)\underline{i} + 2y\underline{j} - 2x\underline{k} = -2\underline{i} + 6\underline{j} - 2\underline{k}$ at $(1,3,2)$. $\underline{n}_2 = \nabla(3x^2-y^2+3z)$ $= 6x\underline{i} - 2y\underline{j} + 3\underline{k} = 6\underline{i} - 6\underline{j} + 3\underline{k}$. So $\underline{s} = \begin{pmatrix}-2\\6\\-2\end{pmatrix}\wedge\begin{pmatrix}6\\-6\\3\end{pmatrix} = \begin{pmatrix}6\\-6\\-24\end{pmatrix}$. This is in direction of decreasing z (-24). Unit vector in direction of increasing z is $\frac{1}{\sqrt{18}}(-\underline{i} + \underline{j} + 4\underline{k})$ & $\frac{\partial w}{\partial s} = \frac{1}{\sqrt{18}}\begin{pmatrix}-1\\1\\4\end{pmatrix}\cdot\begin{pmatrix}3\\6\\3\end{pmatrix} = \frac{15}{\sqrt{18}} = 5/\sqrt2$

d) $\underline{s} = \begin{pmatrix}3\\6\\2\end{pmatrix}$, $\hat{s} = \frac{1}{\sqrt{9+36+4}}\begin{pmatrix}3\\6\\2\end{pmatrix} = \frac17\begin{pmatrix}3\\6\\2\end{pmatrix}$ & $\frac{\partial}{\partial s} = \frac17\left(3\frac{\partial}{\partial x} + 6\frac{\partial}{\partial y} + 2\frac{\partial}{\partial z}\right)$

$f = xy+yz+zx$

& $f_s = \frac37(y+z) + \frac67(x+z) + \frac27(y+x)$. $f_{ss} = \frac{1}{49}\left(3(6+2) + 6(3+2) + 2(3+6)\right)$ $= \frac{1}{49}(24+30+18) = \frac{72}{49}$, $f_{sss} = 0$ (f is a quadratic function). At $(1,1,2)$ the derivatives are $\frac37(1+2) + \frac67(1+2) + \frac27(1+1) = \frac17(9+18+4) = 31/7$, $\frac{72}{49}$, 0

2 a) $z = 2x+4y-x^2-y^2-3$, $z_x = 2-2x$, $z_y = 4-2y$. Both zero at $(1,2)$

$z_{xy} = 0$, $z_{xx} = -2$, $z_{yy} = -2$ & $\Delta = z_{xx}z_{yy} - z_{xy}^2 = (-2)(-2) - 0 = 4 > 0$. $z_{xx} < 0 \Rightarrow$ minimum.

b) $z = x^3+y^3-3xy$, $z_x = 3x^2-3y$, $z_y = 3y^2-3x$. Both zero when $x^2 = y$ & $y^2 = x \Rightarrow$ $x^4 = x$ & $x = 0,1$ giving $y = 0,1$ ie at $(0,0)$ & $(1,1)$. $z_{xx} = 6x$, $z_{yy} = 6y$, $z_{xy} = -3$ & $\Delta = 36xy - 9$. At $(0,0)$, $\Delta = -9 \Rightarrow$ saddle pt. At $(1,1)$, $\Delta > 0$ & $z_{xx} = 6 > 0 \Rightarrow$ minimum

c) $z = (3-2x^2-y^2)(x-y)$, $z_x = -4x(x-y) + 3 - 2x^2 - y^2$, $z_y = -2y(x-y) - 3 + 2x^2 + y^2$.
Setting both to zero & adding gives $(-4x-2y)(x-y) = 0 \Rightarrow x = y$ or $y = -2x$.
Using these in turn in $z_x = 0$ gives $3 = 3x^2 \Rightarrow x = \pm1$ giving $(1,1)$ (A), $(-1,-1)$ (B),
& $-4x(3x) + 3 - 2x^2 - 4x^2 = 0 \Rightarrow 18x^2 = 3$, $x = \pm 1/\sqrt6$, $y = \mp 2/\sqrt6$ ie $(1/\sqrt6, -\frac{2}{\sqrt6})$ (C), $(-\frac{1}{\sqrt6}, \frac{2}{\sqrt6})$ (D)

$z_{xx} = -8x - 4x + 4y = 4y - 12x$, $z_{xy} = 4x - 2y$, $z_{yy} = 6y - 2x$

$\Delta = (4y-12x)(6y-2x) - (4x-2y)^2 = 4[(2y-6x)(3y-x) - (2x-y)^2]$

At (A) $(1,1)$, $\Delta = 4[(-4)(2) - 1^2] < 0 \Rightarrow$ saddle pt.
At (B) $(-1,-1)$ $\Delta = 4[(4)(-2) - (-1)^2] < 0 \Rightarrow$ saddle pt

At (C) $\left(\frac{1}{\sqrt6}, \frac{-2}{\sqrt6}\right)$, $\Delta = \frac{4}{6}\left[(-10)(-7) - 4^2\right] > 0$, $z_{xx} = \frac{1}{\sqrt6}(-8-12) < 0 \Rightarrow$ Maximum

At (D) $\left(-\frac{1}{\sqrt6}, \frac{2}{\sqrt6}\right)$, $\Delta = \frac{4}{6}\left[(10)(7) - (-4)^2\right] > 0$, $z_{xx} = \frac{1}{\sqrt6}(8+12) > 0 \Rightarrow$ Minimum

a) $z = (x^2-1)e^{y^2} - 2x^2$, $z_x = 2x(e^{y^2}-2)$, $z_y = 2y(x^2-1)e^{y^2}$. Setting these to zero gives $x = 0$ or $y = \pm\sqrt{\ln 2}$ & $x = \pm 1$ or $y = 0$. So we have turning pts at $(0,0)$, $(\pm1, \pm\sqrt{\ln 2})$ (all 4 combinations). $z_{xx} = 2(e^{y^2}-2)$, $z_{yy} = e^{y^2}2(x^2-1) + 4y^2(x^2-1)e^{y^2} = e^{y^2}(x^2-1)(2+4y^2)$, $z_{xy} = 4xy\,e^{y^2}$

At $(0,0)$, $\Delta = (-2)(-2) - 0^2 > 0$, $z_{xx} = 2(-1) < 0 \Rightarrow$ Maximum.

At $(\pm1, \pm\sqrt{\ln 2})$, $z_{xx} = z_{yy} = 0$, $\Delta = -z_{xy}^2 < 0 \Rightarrow$ saddle points

3) $f(x,y) = \sin(xy)$. Write $x = 1 + (x-1)$, $y = \pi/2 + (y - \pi/2)$ & use $a = 1$, $b = \pi/2$, $h = (x-1)$, $k = y - \pi/2$ in Taylor Series formula:

$f(a+h, b+k) = f(a,b) + (\underline{h}\cdot\nabla)f|_{a,b} + \frac12(h\cdot\nabla)^2 f|_{(a,b)} + \dots$

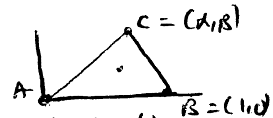
$f(1,\pi/2) = \sin\pi/2 = \underline{1}$, $f_x = y\cos(xy)$; $f_x(1,\pi/2) = \pi/2\cos\pi/2 = \underline{0}$, $f_y = x\cos(xy)$

$f_y(1,\pi/2) = 1\cdot\cos\pi/2 = \underline{0}$, $f_{xx} = -y^2\sin(xy)$, $f_{xx}(1,\pi/2) = -\underline{\pi^2/4}$, $f_{xy} = \cos(xy) - xy\sin(xy)$

$f_{xy}(1,\pi/2) = 0 - 1\cdot\pi/2\cdot\sin\pi/2 = -\underline{\pi/2}$, $f_{yy} = -x^2\sin(xy)$ & $f_{yy}(1,\pi/2) = \underline{-1}$.

$(h\cdot\nabla)f = (x-1)f_x + (y-\pi/2)f_y = 0 + 0$, $(h\cdot\nabla)^2 f = (x-1)^2 f_{xx} + 2(x-1)(y-\pi/2)f_{xy} + (y-\pi/2)^2 f_{yy}$
$= -\pi^2/4\,(x-1)^2 + 2(-\pi/2)(x-1)(y-\pi/2) - (y-\pi/2)^2$.

$f(x,y) = \underline{1 - \pi^2/8\,(x-1)^2 - \pi/2\,(x-1)(y-\pi/2) - \frac12(y-\pi/2)^2}$.

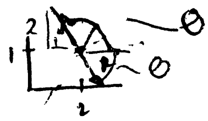


5) wlog set vertices of Δ at $(0,0)$, $(1,0)$ & (α,β). If P is (x,y) then value to be minimised is $f = (x^2+y^2) + [(x-1)^2 + y^2] + [(x-\alpha)^2 + (y-\beta)^2]$ $\partial f/\partial x = 0$
$\Rightarrow 2x + 2(x-1) + 2(x-\alpha) = 0 \Rightarrow x = \frac13(1+\alpha)$. $\partial f/\partial y = 0 \Rightarrow 2y + 2y + 2(y-\beta) = 0 \Rightarrow y = \beta/3$
$f_{xx} = 6$, $f_{yy} = 6$, $f_{xy} = 0 \Rightarrow \Delta > 0$ & $f_{xx} < 0 \Rightarrow$ this is a minimum. $\vec{OP} = \frac13(\vec{OA} + \vec{OB} + \vec{OC})$

4) $T(x,y) = 100xy/(x^2+y^2)$ $\nabla T = 100\begin{pmatrix} y/(x^2+y^2) - \frac{2x^2y}{(x^2+y^2)^2} \\ x/(x^2+y^2) - \frac{2xy^2}{(x^2+y^2)^2}\end{pmatrix} = \frac{100}{(x^2+y^2)^2}\begin{pmatrix} y(y^2-x^2) \\ -x(y^2-x^2)\end{pmatrix}$

a) $\angle\pi/3$ $\hat{s} = \begin{pmatrix}\cos\pi/3 \\ \sin\pi/3\end{pmatrix} = \begin{matrix}1/2 \\ \sqrt3/2\end{matrix}$ $\left.\frac{\partial T}{\partial s}\right|_{2,1} = \frac{100}{(4+1)^2}\begin{pmatrix}1/2 \\ \sqrt3/2\end{pmatrix}\cdot\begin{pmatrix}1(1-4) \\ -2(1-4)\end{pmatrix} = 2\begin{pmatrix}1 \\ \sqrt3\end{pmatrix}\cdot\begin{pmatrix}-3 \\ 6\end{pmatrix}$
$= \overline{12\sqrt3 - 6}$.

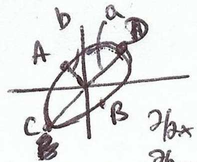
b) Maximum derivative will be in direction of ∇T, angle $\theta = \tan^{-1}\left(\frac{-x}{y}\right)$ with x axis $= \tan^{-1}\left(\frac{-2}{1}\right)$ or $-\tan^{-1}(2)$



in the direction of vector $\begin{pmatrix}1 \\ -2\end{pmatrix}$ or $\frac{1}{\sqrt5}\begin{pmatrix}1 \\ -2\end{pmatrix}$

c) Max value is $\frac{1}{\sqrt5}\begin{pmatrix}1 \\ -2\end{pmatrix}\cdot\frac{100}{25}\begin{pmatrix}1\cdot(-3) \\ -2\cdot(-3)\end{pmatrix} = -3\cdot\frac{100}{25}\cdot\frac{1}{\sqrt5}\,(1^2+2^2)$
$= -12/\sqrt5$.

Sheet 2

1) 

~~Maximise~~/Minimise x^2+y^2, subject to $17x^2-30xy+17x^2=32$. Use Lag multipliers on $(x^2+y^2)-\lambda(17x^2-30xy+17y^2)$. Derivatives being zero give

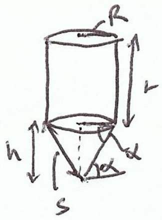
$\partial/\partial x \quad 2x-\lambda(34x-30y)=0$
$\partial/\partial y \quad 2y-\lambda(-30x+34y)=0$ $\Rightarrow \begin{pmatrix} 2-34\lambda & 30\lambda \\ 30\lambda & 2-34\lambda \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}=0$. Non-zero solutions

for x,y exist if $(2-34\lambda)^2=(30\lambda)^2 \Rightarrow 2-34\lambda=\pm 30\lambda \Rightarrow$ $(+)$, $\lambda=2/64=1/32$
$(-)$ $\lambda=2/4 \Rightarrow 1/2$

If $\lambda=1/32$ then, looking at first row $\left[2-34\lambda=\frac{2.32-34}{32}\right]$ $\cancel{30}/_{32}x+30/_{32}y=0 \Rightarrow y=-x$ & from constraint with $-x=y$,

$\cancel{32}\,64\,x^2=32 \Rightarrow x=\pm\cancel{1}/\sqrt{2}$, $y=\mp 1/\sqrt{2}$. These are points A & B & $b=\sqrt{x^2+y^2}=\cancel{\;}=\underline{1}$. If $\lambda=1/2$

then, as before, $-15x+15y=0 \Rightarrow$ & $y=x$, giving $4x^2=32$, $x=\pm 2\sqrt{2}$, $y=\pm 2\sqrt{2}$. These are points C & D & $a=\sqrt{x^2+y^2}=\underline{4}$. Area is $\pi ab=\pi.4.\cancel{\;}=\cancel{\;}\;\underline{\underline{4\pi}}$.

2) 

Volume of cone is $\frac{1}{3}\pi R^2 h$ (with $h/R=\tan\alpha$) $=\frac{1}{3}\pi R^3\tan\alpha$
Volume of cylinder is $\pi R^2 L$ $\Bigg\}\Rightarrow$ Volume
Area of curved part of cone is $\pi R s$ (with $s/R=\sin\alpha$) $=\pi R^2\sec\alpha$
Area of " cylinder is $2\pi R L$ $\Bigg\}\Rightarrow$ Area.

Use Lagrange Multipliers on $\left(\frac{1}{3}\pi R^3\tan\alpha+\pi R^2 L\right)-\lambda\left(2\pi R L+\pi R^2\sec\alpha\right)=f(R,L,\alpha)$

$\partial f/\partial R=0 \Rightarrow \pi R^2\tan\alpha+2\pi R L-\lambda 2\pi L-\lambda 2\pi R\sec\alpha=0$ (A)
$\partial f/\partial L=0 \Rightarrow \pi R^2-2\pi R\lambda=0$ (B) $\Rightarrow R=2\lambda$, discounting $R=0$, $\lambda=R/2$
$\partial f/\partial\alpha=0 \Rightarrow \frac{1}{3}\pi R^3\sec^2\alpha-\cancel{2}\pi R^2\lambda\sec\alpha\tan\alpha=0$ (C) $\Rightarrow \frac{R}{3}=\lambda\tan\alpha\cos\alpha=\lambda\sin\alpha$ discounting $\alpha=\pi/2$.

Now B & C give $\underline{\sin\alpha=2/3}$, discounting $\lambda=0$ which leads to $R=0$, so $\sec\alpha=3/\sqrt{5}$, $\tan\alpha=2/\sqrt{5}$

(A) $\Rightarrow L=\frac{1}{2\pi(R-\lambda)}\left\{2\pi\lambda R\sec\alpha-\pi R^2\tan\alpha\right\}=\frac{1}{2\pi R/2}\left\{2\pi\frac{R^2}{2}.\frac{3}{\sqrt{5}}-\pi R^2.\frac{2}{\sqrt{5}}\right\}=R/\sqrt{5}$

Using the constraint now, $\pi R^2\sec\alpha+2\pi R L=S$ with $L=R/\sqrt{5}$, gives $\pi R^2\frac{3}{\sqrt{5}}+2\pi\frac{R^2}{\sqrt{5}}=S$

gives $R^2=S/\sqrt{5}\pi$, $\underline{R=S^{1/2}.5^{-1/4}\pi^{-1/2}}$, $\underline{L=S^{1/2}5^{-3/4}\pi^{-1/2}}$, $\underline{\sin\alpha=2/3}$ and $V=\frac{1}{3}\pi S^{3/2}5^{-3/4}\pi^{-3/2}\frac{2}{\sqrt{5}}$

$+\pi S 5^{-1/2}\pi^{-1}S^{1/2}5^{-3/4}\pi^{-1/2}=\frac{S^{3/2}}{\pi^{1/2}5^{1/4}}\left\{\frac{2}{3.5}+\frac{1}{5}\right\}=\underline{S^{3/2}/3\pi^{1/2}5^{1/4}}$. Minimum achieved for $\underline{R=0}$.

~~with~~.

3) Square of distance from point (x,y,z) to (x_0,y_0,z_0) is $(x-x_0)^2+(y-y_0)^2+(z-z_0)^2$. Use Lagrange multipliers to find extremum of this subject to the constraint $ax+by+cz=-d$. Consider $[(x-x_0)^2+(y-y_0)^2+(z-z_0)^2]-\lambda(ax+by+cz)=f(x,y,z,\lambda)$ & set $f_x=f_y=f_z=0$ to give $2(x-x_0)=a\lambda$, $2(y-y_0)=b\lambda$, $2(z-z_0)=c\lambda$. Substituting solutions to these for x,y,z into constraint gives $a[x_0+a\lambda/2]+b[y_0+b\lambda/2]+c[z_0+c\lambda/2]=-d$ gives

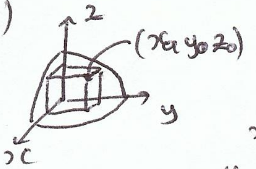
$\lambda=-2[d+(ax_0+by_0+cz_0)]/(a^2+b^2+c^2)$ & $x=x_0-a[d+(ax_0+by_0+cz_0)]/(a^2+b^2+c^2)$
$y=y_0-b[\quad]/(\quad)$
(Can you explain this in terms of vectors as in M1401) $z=z_0-c[\quad]/(\quad)$

$(\text{Distance})^2$ is $(x-x_0)^2+(y-y_0)^2+(z-z_0)^2=\frac{a^2\lambda^2}{4}+\frac{b^2\lambda^2}{4}+\frac{c^2\lambda^2}{4}=\frac{(a^2+b^2+c^2)}{4}.4\frac{[d+ax_0+by_0+cz_0]^2}{(a^2+b^2+c^2)^2}$

& taking $\sqrt{\;}$ gives the result.

4) Consider $xy^2z^3-\lambda(x+y+z)=g(x,y,z)$. Set $g_x=g_y=g_z=0$ to give, in turn,
$\underset{(A)}{\lambda=y^2z^3}$, $\underset{(B)}{\lambda=2xyz^3}$, $\underset{(C)}{\lambda=3xy^2z^2}$. Taking ratios of these $A/B \Rightarrow 1=y/2x \Rightarrow y=2x$
$C/B \Rightarrow 1=3y/2z \Rightarrow z=3y/2=3x$

Constraint now gives $x+y+z=x+2x+3x=6 \Rightarrow x=1$, $y=2$, $z=3$ & extreme value is $1.2^2.3^3=\underline{108}$. This is a max. as a min. is obtained with $x+y=6$ & $z=0$, say

5) 

Let parallelepiped have a corner at (x_0, y_0, z_0). The volume is 8… & x_0, y_0, z_0 can be found by finding maximum of $8xyz$ subject to $x^2/9 + y^2/16 + z^2/36 = 1$. Consider $8xyz - (x^2/9 + y^2/16 + z^2/36)$ & setting first partial derivatives to zero, gives $\underset{(A)}{8yz = 2x/9}$, $\underset{(B)}{8xz = 2y/16}$, $\underset{(C)}{8xy = 2z/36}$

$\Rightarrow$ A/B $\Rightarrow$ $y/x = \frac{16}{9}\frac{x}{y}$ $\Rightarrow$ $y^2 = \frac{16}{9}x^2$, A/C $\Rightarrow$ $z/x = \frac{36}{9}\frac{x}{z}$ $\Rightarrow$ $z^2 = \frac{36}{9}x^2$ (we are not interested in zero values of x, y, z

Substitution into constraint gives $3x^2/9 = 1$ $\Rightarrow$ $x = \sqrt{3}$, $y = \frac{4}{3}\sqrt{3}$, $z = \frac{6}{3}\sqrt{3}$ – wlog $x, y, z > 0$

& volume is $8 \cdot \frac{4}{\cancel{3}} \cdot \frac{\cancel{6}^{\,2}}{\cancel{3}} \cdot \cancel{3}\sqrt{3} = \underline{64\sqrt{3}}$

6) Maximise/Minimize z subject to constraint $2x^2 + 3y^2 + z^2 - 12xy + 4xz = 35$. Consider $g(x,y,z) = z - \lambda(2x^2 + 3y^2 + z^2 - 12xy + 4xz)$ & set $g_x = g_y = g_z = 0$ to give

$\underset{(A)}{-\lambda(4x - 12y + 4z) = 0}$, $\underset{(B)}{-\lambda(6y - 12x) = 0}$, $\underset{(C)}{1 = \lambda(2z + 4x)}$. (C) $\Rightarrow \lambda \neq 0$ & thus

(B) $\Rightarrow$ $y = 2x$ & $z = 3y - x = 5x$. Put these in constraint to find z: $2(z^2/25) + 3 \cdot \frac{4z^2}{25} + z^2 - 12 \cdot \frac{2z^2}{25} + 4z^2/5 = 35$

$\Rightarrow$ $x = z/5$, $y = 2z/5$

$\Rightarrow$ $z^2[2 + 12 + 25 - 24 + 20] = \cancel{35} \cdot 25$ $\Rightarrow$ $\underline{z = \pm 5}$.

7) As for Q 3 we need to find points that max/min $(x-1)^2 + (y-2)^2 + (z-2)^2$ subject to $x^2 + y^2 + z^2 = 1$

Consider $g(x,y,z) = (x-1)^2 + (y-2)^2 + (z-2)^2 - \lambda(x^2 + y^2 + z^2)$ & set $g_x = g_y = g_z$ to zero to give $2(x-1) = 2\lambda x$ $2(y-2) = 2\lambda y$ $2(z-2) = 2\lambda z$ $\Rightarrow$ $x = 2/(2-2\lambda) = 1/(1-\lambda)$, $y = 2/(1-\lambda)$, $z = \frac{2}{1-\lambda}$

& the constraint gives $1 + 4 + 4 = (1-\lambda)^2$ $\Rightarrow$ $1 - \lambda = \pm 3$, $\lambda = 4$ or $\lambda = -2$.

$\lambda = 4$ gives $(x,y,z) = (-1/3, -2/3, -2/3)$ & $(\text{distance})^2$ $16/9 + 64/9 + 64/9$ $\Rightarrow$ biggest

$\lambda = -2$ gives $(x,y,z) = (1/3, 2/3, 2/3)$ & $(\text{distance})^2$ $4/9 + 16/9 + 16/9$ $\Rightarrow$ smallest.

Sheet 3

1) $F(x,y,y') = y'^2 + y^2 + 2xy$. Eulers eqn is $0 = \frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = (2y+2x) - \frac{d}{dx}(2y') \Rightarrow y'' - y = x$, ⊕ with solution $y = Ae^{x} + Be^{-x} - x$. If $y(0)=0$, $y(1)=-1$, then $A=B=0$ & $\underline{y=-x}$. If $y = f+g$ then

$I(f+g) = \int_0^1 (f+g)'^2 + (f+g)^2 - 2x(f+g)\,dx$ with $f'' - f = x$ & $f = -x$. ①

$I(f) = \int_0^1 f'^2 + f^2 - 2x(f)\,dx$. $I(f+g) - I(f) = \int_0^1 [(f+g)'^2 - f'^2] + [(f+g)^2 - f^2] + 2xg\,dx$

$= \int_0^1 f'^2 + 2f'g' + f^2 + 2fg + 2xg\,dx$ Ⓐ. Taking ①, multiplying by $2g$ & integrating gives Ⓑ

$\int_0^1 2f''g - 2fg - 2xg\,dx = \underbrace{[2f'g]_0^1}_{0 \text{ as } g(0)=g(1)=0} - \int_0^1 2f'g' + 2fg + 2xg\,dx = 0$. Adding Ⓐ&Ⓑ gives $I(f+g) - I(f)$

$= \int_0^1 f'^2 + f^2\,dx$ which is positive. Hence f minimizes integral

2) a) $\int_1^2 \underbrace{\frac{y'^2}{x^3}}_{F}\,dx$, $y(1)=2$, $y(2)=17$ $0 = \frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0 - \frac{d}{dx}\left[\frac{2y'}{x^3}\right] \Rightarrow \left[\frac{2y'}{x^3}\right]' = 0$ & so $y' = Cx^{3/2}$ for

some constant. $y = Cx^4/8 + D$ ~~the b.c.'s give~~ $C = $ B.C's $\Rightarrow$ $y(1) = 2 \Rightarrow 2 = C/8 + D$, $y(2) = 17 \Rightarrow 17 = C\,16/8 + D$ $\Big\}$ $15 = \frac{15C}{8}$, $C = 8$, $D = 1$

$\Rightarrow \underline{y = 1 + x^4}$, Minimum as $\int_1^2 \frac{y'^2}{x^3}\,dx$ can be made as large as we like

b) $\int_0^{\pi/2} y^2 - y'^2 - 2y\sin x\,dx$ $y(0)=1$, $y(\pi/2)=2$; $0 = \frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 2y - 2\sin x - \frac{d}{dx}(-2y') \Rightarrow y'' + y = \sin x$

$\Rightarrow y = A\sin x + B\cos x + \alpha x\cos x$ with $(0 + 2\alpha(-\sin x)) + \alpha x(-\cos x) + \alpha x\cos x = \sin x$, $\alpha = -\frac{1}{2}$.

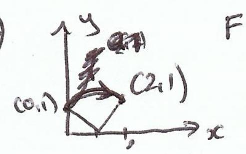
BC $\Rightarrow y(0) = B = 1$ & $y(\pi/2) = A = 2$, so $y = \underline{2\sin x + \cos x - \frac{1}{2}x\cos x}$. Maximum

as $-y'^2$ can be made as large & negative as we like

c) $\int_0^{\pi} y'^2 + 2y\sin x\,dx$, $y(0) = y(\pi) = 0$, $0 = \frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 2\sin x - \frac{d}{dx}(2y') \Rightarrow y'' = \sin x$

$\Rightarrow y = -\sin x + Ax + B$. BC $\Rightarrow 0 = 0 + B$ &, $0 = 0 + A\pi + B \Rightarrow A = B = 0$ & $\underline{y = -\sin x}$

Minimum as y'^2 can be as large & +ve as we like

3) 

$F[x,y,y'] = \frac{(1+y'^2)^{1/2}}{y}$ ⇒

no $x \Rightarrow F - y'\frac{\partial F}{\partial y'} = \text{Const}$

$\Bigg\}$ $\frac{(1+y'^2)^{1/2}}{y} - \frac{y'}{y}\frac{y'}{(1+y'^2)^{1/2}} = \left[1 + y'^2 - y'^2\right]\frac{1}{y(1+y'^2)^{1/2}} = \frac{-1}{y(1+y'^2)^{1/2}} = \text{Const}$

$\Rightarrow$ ⊛ $\underline{y\left[1+y'^2\right]^{1/2} = \text{Constant} = C}$

Solve for y' so as to integrate $1 + y'^2 = \frac{C^2}{y^2} \Rightarrow y'^2 = \frac{C^2 - y^2}{y^2} \Rightarrow \frac{dy}{dx} = \pm\frac{\sqrt{C^2-y^2}}{y}$ &

$\int_0^x dx = \pm\int_1^y \frac{y}{\sqrt{C^2-y^2}}\,dy \Rightarrow x = \pm\left[-\sqrt{C^2-y^2}\right]_1^y = \pm\left\{\sqrt{C^2-1} - \sqrt{C^2-y^2}\right\}$ using bc $y(0)=1$

The curve must also pass through $(2,1)$. it has the form $x \pm \sqrt{C^2-1} = \pm\sqrt{C^2-y^2}$

$\Rightarrow (x-B)^2 = C^2 - y^2$ or $(x-B)^2 + y^2 = C^2$, A circle radius C & centre $(B,0)$, $B = \pm\sqrt{C^2-1}$

Using other b.c. $(2-B)^2 + 1 = C^2 \Rightarrow 4 - 4B + \underbrace{(C^2-1)}_{B^2=C^2-1} + 1 = C^2 \Rightarrow B = 1 \Rightarrow C = \sqrt{2}$ & $(x-1)^2 + y^2 = 2$

The time is $\frac{1}{\lambda}\int_0^2 \frac{(1+y'^2)^{1/2}}{y}\,dx = \frac{1}{\lambda}\int_0^2 \frac{1}{y}\frac{C}{y}\,dx$, from ⊛ & the C is the same one & $y^2 = C^2 - (x-1)^2$

$T = \frac{\sqrt{2}}{\lambda}\int_0^2 \frac{dx}{2-(x-1)^2} = \frac{\sqrt{2}}{\lambda}\int_0^2 \left\{\frac{1}{\sqrt{2}-(x-1)} + \frac{1}{\sqrt{2}+(x-1)}\right\}\frac{dx}{2(\sqrt{2})} = \frac{\sqrt{2}}{2\lambda\sqrt{2}}\left[\ln\frac{(-1+\sqrt{2}+x)}{(1+\sqrt{2}-x)}\right]_0^2$

taking +ve root, $y > 0$

$= \frac{1}{2\lambda}\ln\left\{\frac{(1+\sqrt{2})(1+\sqrt{2})}{(\sqrt{2}-1)(\sqrt{2}-1)}\right\} = \frac{2}{\lambda}\ln(1+\sqrt{2})$, $\left\{\begin{array}{l}(\sqrt{2}-1)(\sqrt{2}+1) = 2-1 = 1 \\ \text{so } \frac{1}{\sqrt{2}-1} = \sqrt{2}+1\end{array}\right\}$

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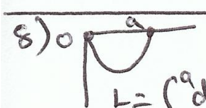
8) 

$L = \int_0^a ds = \int_0^a \sqrt{1+y'^2}\,dx$

Use Lagrange multipliers on $\int_0^a y\sqrt{1+y'^2}\,dx$ with constraint $\int_0^a \sqrt{1+y'^2}\,dx = L$ & consider

$\int_0^a y(1+y'^2)^{1/2} - \lambda(1+y'^2)^{1/2}\,dx = \int_0^a (y-\lambda)(1+y'^2)^{1/2}\,dx$

Beltrami's eqn is $(y-\lambda)(1+y'^2)^{1/2} - (y-\lambda)\frac{y'^2}{(1+y'^2)^{1/2}} = \frac{(y-\lambda)}{(1+y'^2)^{1/2}} = C$

PTO

4) Use $F - y'\dfrac{\partial F}{\partial y'} = C$

a) $F = \dfrac{y'^2}{1+y^2} \Rightarrow \dfrac{y'^2}{1+y^2} - y'\cdot\dfrac{2y'}{1+y^2} = \dfrac{-y'^2}{1+y^2} = C \Rightarrow +\displaystyle\int\frac{dy}{\sqrt{1+y^2}} = \int \tilde{C}dx \Rightarrow +\sinh^{-1}y = \tilde{C}x + D$,

$y(0)=0 \Rightarrow D=0$, $y(1)=2 \Rightarrow +\sinh^{-1}2 = \tilde{C}$ & $\cancel{\pm}\sinh^{-1}y = \cancel{\pm}\sinh^{-1}2x$, $\underline{y = \sinh(\sinh^{-1}2\,x)}$

b) $F = y'^2/y^3 \Rightarrow y'^2/y^3 - y'\dfrac{2y'}{y^3} = -y'^2/y^3 = C \Rightarrow \displaystyle\int\frac{dy}{y^{3/2}} = \int dx\,\tilde{C}$, $\tilde{C} = -C^{1/2} \Rightarrow -2y^{-1/2} = \tilde{C}x + D$

$y(0) = 1 \Rightarrow -2 = D$, $y(2) = 4 \Rightarrow -2/2 = \tilde{C}\,2 + D \Rightarrow -1 = 2\tilde{C} - 2 \Rightarrow \tilde{C} = 1/2$ &

$y^{-1/2} = 1 - x/4$, $\underline{y = (1-x/4)^{-2}}$.

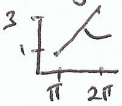
c) $F = \frac{1}{2}y'^2 + yy' + y + y' \Rightarrow \frac{1}{2}y'^2 + \cancel{yy'} + y + \cancel{y'} - y'\left(\frac{2}{2}y' + \cancel{y} + \cancel{1}\right) = C \Rightarrow \cancel{y'^2/y} \; y'^2/2 = C + y$

$\Rightarrow \pm\frac{1}{\sqrt{2}}\displaystyle\int\frac{dy}{\sqrt{C+y}} = \int dx \Rightarrow \pm\sqrt{C+y} = \frac{x+B}{\sqrt{2}}$, $y = \dfrac{(x+B)^2}{2} - C$, $y(0)=0 \Rightarrow C = B^2/2 \Rightarrow y = \frac{x^2}{2} + xB$

$y(2) = 2 \Rightarrow 2 = \frac{4+4B}{2} + \frac{B^2}{2} - \cancel{C} \Rightarrow B = 0$

$\underline{y = \frac{1}{2}x^2}$

5) $\displaystyle\int_\pi^{2\pi} \underbrace{y'^2[1+y'^2]^{3/2}}_{F}\,dx$, $\underbrace{\dfrac{\partial F}{\partial y}}_{=0\text{ have}} - \dfrac{d}{dx}\left(\dfrac{\partial F}{\partial y'}\right) = 0 \Rightarrow \partial F/\partial y' = \text{Const}$, but F depends only on $y' \Rightarrow$ some function of $y' = \text{const} \Rightarrow y' = \text{const} = 2/\pi$ from bc.



$I = \displaystyle\int_\pi^{2\pi}\left(\frac{4}{\pi^2}\right)\left[1+\frac{4}{\pi^2}\right]^{3/2}dx = \pi\cdot\frac{4}{\pi^2}\left[1+\frac{4}{\pi^2}\right]^{3/2} = 4(\pi^2+4)^{3/2}/\pi^4$

6) $I(y) = \frac{1}{2}\displaystyle\int_0^1 y'^2 + y^2\,dx$. EL eqn is $y - \frac{d}{dx}y' = 0 \Rightarrow y'' = y$ for any extremal y. (not true as $f+\varphi$ is not extremal)

Use this & integration by parts on the y'^2 term. $I(\varphi) = \frac{1}{2}\displaystyle\int_0^1 \varphi'^2 + \varphi^2\,dx$

$= \left[\frac{1}{2}\varphi'\varphi\right]_0^1 - \frac{1}{2}\displaystyle\int_0^1 \underbrace{\varphi\varphi'' - \varphi^2}_{=0\text{ as }\varphi''=\varphi}dx = \frac{1}{2}\left[\varphi(1)\varphi'(1) - \varphi(0)\varphi'(0)\right]$. $\cancel{I[\varphi+f] = \frac{1}{2}\left[[\varphi(1)+f(1)][\varphi'(1)+f'(1)] - [\varphi(0)+f(0)][\varphi'(0)+f'(0)]\right]}$

$\cancel{= I(\varphi) + \frac{1}{2}\left(\varphi(1)f'(1) - \varphi(0)f'(0)\right)}$

Consider $I(f+\varphi) - I(\varphi) = \frac{1}{2}\displaystyle\int_0^1 (f'^2 + \underline{2f'\varphi'}) + (f^2 + \underset{\sim}{2f\varphi})\,dx$ & $\varphi'' = \varphi$ for extremal φ.

The term $\sim$ gives, on integrating by parts $\underbrace{[f\varphi']_0^1}_{=0\text{ as } f(0)=f(1)=0} - \frac{1}{2}\displaystyle\int_0^1 f\varphi''dx - \frac{1}{2}\int_0^1 f\varphi\,dx$ from this, cancelling $\sim$

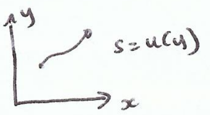
$\Rightarrow I(f+\varphi) - I(\varphi) = \frac{1}{2}\displaystyle\int_0^1 f'^2 + f^2\,dx > 0 \Rightarrow$ minimising

$\varphi'' = \varphi \Rightarrow \varphi = Ae^x + Be^{-x}$, $\varphi(0) = 1 \Rightarrow \underset{①}{A+B=1}$, $\varphi(1) = 1 \Rightarrow Ae + B/e = 1$ ②

$I(\varphi) = \frac{1}{2}[\varphi(1)\varphi'(1) - \varphi(0)\varphi'(0)] = \frac{1}{2}[\varphi'(1) - \varphi'(0)] = \frac{1}{2}\left[\left(Ae - \frac{B}{e}\right) - (A - B)\right]$

$= \frac{1}{2}\left[A(e-1) + B(1-1/e)\right] = \frac{1}{2}\cdot 2A(e-1)$ as ① ② $\Rightarrow A(e-1) + B(1/e - 1) = 0 \Rightarrow B(1-1/e) = A(e-1)$

① $- e$② $\Rightarrow A(1-e^2) + 0B = 1-e \Rightarrow A = (1-e)/(1-e)(1+e) = 1/1+e \Rightarrow I = (e-1)/(e+1)$

7) 

$T = \displaystyle\int dt = \int\frac{dt}{ds}ds = \int\frac{\sqrt{1+y'^2}dx}{u(y)}$ ($ds/dt = u$) & EL eqn is (Beltrami) $\dfrac{(1+y'^2)^{1/2}}{u} - y'\cdot\dfrac{1}{u}\dfrac{y'}{(1+y'^2)^{1/2}} = \text{Const}, C$

$\Rightarrow \dfrac{1}{(1+y'^2)^{1/2}}\left[(1+y'^2) - y'^2\right] = Cu$. Solve for y':

$1 + y'^2 = \dfrac{1}{c^2u^2} \Rightarrow \dfrac{dy}{dx} = \pm\left(\dfrac{1-c^2u^2}{c^2u^2}\right)^{1/2} = \left(\dfrac{A^2-u^2}{u^2}\right)^{1/2}$, $C^2 = 1/A^2$

$\Rightarrow x - B = \pm\displaystyle\int\frac{u}{\sqrt{A^2-u^2}}dy$

8cld $\Rightarrow \left(\dfrac{dy}{dx}\right)^2 = \dfrac{(y-\lambda)^2}{c^2} - 1 = k^2(y+h)^2 - 1$ with $k^2 = 1/c^2, \lambda = -h$

~~8cld~~ Integrating: $\displaystyle\int\frac{dy}{\sqrt{k^2(y+h)^2-1}} = \pm\int dx \Rightarrow \pm(x+b) = \frac{1}{k}\cosh^{-1}(k(y+h)) \Rightarrow y = -h + \frac{1}{k}\cosh(kx+\alpha)$

as $\alpha = kb$ & cosh is even. BC $y(0) = y(a) = 0$ gives $\cosh\alpha = \cosh(ka+\alpha) \Rightarrow \alpha = -(ka+\alpha)$ (cosh is even) $\Rightarrow \alpha = -ka/2$,

and $\underline{hk = \cosh\alpha = \cosh(ka/2)}$. A second equation for k & h comes from the constraint:

$L = \displaystyle\int_0^a\sqrt{1+y'^2}dx = \int_0^a\frac{y-\lambda}{c}dx = \int_0^a k(y+h)dx = \int_0^a\cosh(kx+\alpha)dx = \frac{1}{k}\sinh(ka+\alpha) \Rightarrow$

$\underline{kL = \sinh(ka/2)}$

Sheet 4

1. a) $\left.\begin{array}{l} v_t + e^x v_x = 0 \\ v(x,0) = x \end{array}\right\} \Rightarrow$ $dt/dq = 1$, $dx/dq = e^x$, $dv/dq = 0$; when $q = 0$, $x = s$, $t = 0$, $v = s$, using q to trace characteristic $\Rightarrow$ $t = q + t_0$, $q = t$ from bc; $-e^{-x} = q - e^{-s}$, using bc for constant; $v = v_0 \Rightarrow v = s$

Eliminate q & s, $q = t$, $s = -\ln(q + e^{-x}) = -\ln(t + e^{-x})$ $\Rightarrow$ $\underline{v = -\ln(t + e^{-x})}$

b) $\left.\begin{array}{l} u_x - 2u_y = u \\ u(0,y) = y \end{array}\right\} \Rightarrow$ $dx/dt = 1$, $dy/dt = -2$, $du/dt = u$; when $t = 0$, $x = 0$, $y = s$, $u = s$ $\Rightarrow$ $x = t + x_0 \Rightarrow x = t$ from bc; $y = -2t + y_0 \Rightarrow y = s - 2t$; $u = Ae^t \Rightarrow u = se^t$

Eliminate t & s, $t = x$, $s = y + 2x$, $\underline{u = e^x(y + 2x)}$

c) $\left.\begin{array}{l} xu_x + yu_y = 2u \\ u(x,1) = x^2 \end{array}\right\} \Rightarrow$ $dx/dt = x$, $dy/dt = y$, $du/dt = 2u$; when $t = 0$, $x = s$, $y = 1$, $u = s^2$ $\Rightarrow$ $x = Ae^t = se^t$ from bc; $y = Be^t = e^t$ from bc; $u = Ce^{2t} = s^2e^{2t}$ from bc $\Rightarrow$ Eliminate s, t. $\underline{u = x^2}$

d) $\left.\begin{array}{l} yu_x - xu_y = 2xyu \\ u(x,x) = x^2 \end{array}\right\}$ $dx/dt = y$, $dy/dt = -x$, $\frac{du}{dt} = 2xyu$ (*); $t = 0$, $x = s$, $y = s$, $u = s^2$ $\Rightarrow$ $d^2x/dt^2 = dy/dt = -x \Rightarrow x = A\cos t + B\sin t$; $y = dx/dt \Rightarrow y = -A\sin t + B\cos t$

b.c's $\Rightarrow$ $s = A + 0$, $s = 0 + B$ $\Rightarrow$ $x = s(\cos t + \sin t)$, $y = s(\cos t - \sin t)$.

So $\frac{du}{u} = 2s^2(\cos t + \sin t)(\cos t - \sin t)dt = 2s^2(\cos^2 t - \sin^2 t)dt = 2s^2\cos 2t\,dt \Rightarrow \ln u = s^2\sin 2t + \text{Const}$

*$\Rightarrow$ $u = Ce^{s^2\sin 2t} = s^2e^{s^2\sin 2t}$ from bc. Now eliminate s & t: $x + y = 2s\cos t$, $x - y = 2s\sin t$

$\Rightarrow (x+y)^2 + (x-y)^2 = 4s^2 \Rightarrow 4s^2 = 2(x^2 + y^2)$, $s^2 = \frac{1}{2}(x^2 + y^2)$. $\sin 2t = 2\sin t\cos t = \frac{2(x+y)(x-y)}{4s^2}$

$\Rightarrow$ $s^2\sin 2t = \frac{1}{2}(x^2 - y^2)$ & $\underline{u(x,y) = \frac{1}{2}(x^2 + y^2)e^{\frac{1}{2}(x^2 - y^2)}}$

2. a) $(x+y)u_x - (x+y)u_y = u$, $\frac{dx}{x+y} = \frac{-dy}{x+y} = \frac{du}{u} \Rightarrow \frac{dy}{dx} = \frac{-(x+y)}{(x+y)} = -1 \Rightarrow y = -x + c$, $\underline{y + x = \varphi}$

Then $\frac{du}{u} = \frac{dx}{x+y} = \frac{dx}{\varphi} \Rightarrow \ln u = \frac{x}{\varphi} + f(\varphi)$ (φ constant) $\Rightarrow$ $\underline{u = g(\varphi)e^{x/x+y}}$ is x

b) $yu_x + xu_y = 1$, $\frac{dx}{y} = \frac{dy}{x} = \frac{du}{1} \Rightarrow \frac{dy}{dx} = \frac{x}{y} \Rightarrow y\,dy = x\,dx \Rightarrow \frac{y^2}{2} = \frac{x^2}{2} + \text{Const}$ $\quad \underline{y^2 - x^2 = \varphi}$. $y =$

Then $\frac{dx}{y} = du = \pm\frac{dx}{\sqrt{\varphi + x^2}} \Rightarrow u = f(\varphi) \pm \sinh^{-1} x/\sqrt{\varphi} \Rightarrow u = f(y^2 - x^2) \pm \sinh^{-1}\frac{x}{\sqrt{y^2 - x^2}}$

[The + sign is the right solution, but I can't quite figure out how to lose − sign] $\varphi^{1/2} = \pm\sqrt{\varphi}$, sinh odd.

c) $x(x+y)u_x + y(x+y)u_y = -(x-y)(2x + 2y + u)$

$\frac{dx}{x(x+y)} = \frac{dy}{(x+y)y} = \frac{du}{-(x-y)(2x+2y+u)} \Rightarrow \frac{dy}{y} = \frac{dx}{x} \Rightarrow \ln y = \ln x + C \Rightarrow \underline{y/x = \varphi}$, $y = \varphi x$

$\Rightarrow \frac{dx}{x^2(1+\varphi)} = \frac{du}{-x(1-\varphi)(2x(1+\varphi) + u)} \Rightarrow \frac{du}{dx} = \frac{-x(1-\varphi)[2(1+\varphi)x + u]}{x^2(1+\varphi)} = -2(1-\varphi) - \frac{(1-\varphi)}{(1+\varphi)}\frac{u}{x}$

$\Rightarrow \frac{du}{dx} + \frac{u}{x}\frac{(1-\varphi)}{1+\varphi} = -2(1-\varphi)$. Solve this with φ const. The integrating factor is $x^{\frac{1-\varphi}{1+\varphi}}$

$\Rightarrow \frac{d}{dx}\left[ux^{\frac{1-\varphi}{1+\varphi}}\right] = -2(1-\varphi)x^{\frac{(1-\varphi)}{1+\varphi}} \Rightarrow ux^{\frac{1-\varphi}{1+\varphi}} = -2(1-\varphi)\frac{(1+\varphi)}{2}x^{2/1+\varphi} + f(\varphi)$ $\quad \left(\frac{1-\varphi}{1+\varphi} + 1 = \frac{2}{1+\varphi}\right)$

$\Rightarrow u = (\varphi - 1)(\varphi + 1)x + x^{\frac{\varphi - 1}{\varphi + 1}}f(\varphi) = x(y^2/x^2 - 1) + x^{y/x - 1/y/x + 1}f(y/x)$

$= \underline{(y^2 - x^2)/x + x^{\frac{y-x}{y+x}}f(y/x)}$

3. $\left.\begin{array}{l} v_t + cv_x = f(x,t) \\ v(x,0) = F(x) \end{array}\right.$ $\Rightarrow$ $dt/dq = 1$, $dx/dq = c$, $\frac{dv}{dq} = f(x,t)$; at $q = 0$, $t = 0$, $x = s$, $v = F(s)$ $\Rightarrow$ $t = q$ using $t = 0$ at $q = 0$; $x = s + cq$ using $x = s$ at $q = 0$

& then $\frac{dv}{dq} = f(x,t) = f(s + cq, q) \Rightarrow v(q) = F(s) + \int_0^q f(s + c\tilde{q}, \tilde{q})d\tilde{q}$

Now eliminate s & q, $q = t$, $s = x - ct$, $v(x,t) = F(x - ct) + \int_0^t f(x - ct + c\tilde{q}, \tilde{q})d\tilde{q}$

If $f(x,t) = xt$, $F(x) = \sin x$, $v(x,t) = \sin(x - ct) + \int_0^t (x - ct + c\tilde{q})\tilde{q}\,d\tilde{q}$

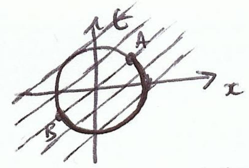
$= \sin(x - ct) + t(x - ct) + ct^3/2$

4) $\frac{dx}{dq} = a(x)$ $\frac{dt}{dq} = b(t)$ $\frac{dv}{dq} = 0$ $\Rightarrow$ $\frac{dx}{a(x)} = dq$; $\frac{dt}{b(t)} = dt$ $\Rightarrow$ $A'dx = dq$, $B'dt = dq$

using q to trace x

$\Rightarrow A'dx - B'dt = 0 \Rightarrow A - B = \text{Const}$ on x

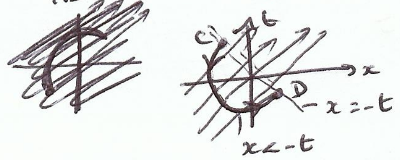
$\Rightarrow$ $\underline{v = F(A - B)}$

5) 

χ eqns are $\frac{dt}{dq}=1$ ①, $\frac{dx}{dq}=1$ ②, $\frac{dv}{dq}=0$ ③. Subtracting ① & ②, $\frac{dt}{dq}$ ...

$\Rightarrow \frac{d}{dt}(t-x)=0 \Rightarrow t-x=$ Const on χ, χ are $t=x+\kappa$. & fro...

~~as~~ v is const on χ. This contradicts the boundary data which says, from $v=x$, $v=1/\sqrt{2}$, at A, $v=-1/\sqrt{2}$ at B, but both these points are on $\chi=0$.



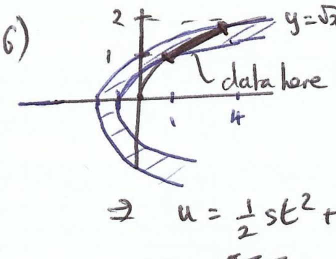
Initial data is, using a parametrisation sensible for the circle, is $x=\cos\theta$, $t=\sin\theta$, $v=\cos\theta$ with $3\pi/4\le\theta\le 7\pi/4$ when $q=0$

$t=q+\sin\theta$, $x=q+\cos\theta$, $v=\cos\theta$. Eliminating θ & q

$v=x-q$ & $(x-q)^2+(t-q)^2=1 \Rightarrow 2q^2-2q(x+t)+(x^2+t^2-1)=0$

$\Rightarrow v=x-\left\{\frac{2(x+t)\pm\sqrt{4(x+t)^2-8(x^2+t^2-1)}}{4}\right\}$. We need to choose $-$ root so $v=x$ on $x^2+t^2=1$ $(q=0)$

This tidies up to $v=\frac{(x-t)}{2}-\frac{1}{2}\sqrt{2-(x-t)^2}$, with a $\sqrt{\ }$, non differentiable singularity at $(x-t)^2=2$, satisfied at points C & D, At C, $x=-1/\sqrt{2}$, $t=1/\sqrt{2}$, $x-t=-\frac{2}{\sqrt{2}}=-\sqrt{2}$, for example.

6) 

χ eqns are $\frac{dx}{dt}=y$ ①, $\frac{dy}{dt}=1$ ②, $\frac{du}{dt}=x$ ③

at $t=0$, $x=s^2$, $y=s$, $u=\frac{2}{3}s^3$

So ② $\Rightarrow y=t+c$, $y=s+t$

① $\frac{dx}{dt}=y=s+t$,

$\Rightarrow x=st+t^2/2+C$, $x=st+t^2/2+s^2$, ③ $\frac{du}{dt}=x=st+t^2/2+s^2$

$\Rightarrow u=\frac{1}{2}st^2+t^3/6+s^2t+\frac{2}{3}s^3$. Eliminate s & t: $y^2=s^2+2st+t^2$, $2x=2s^2+2st+t^2$, so

$s^2=2x-y^2$ & $t=y-s=y\pm\sqrt{2x-y^2}$ & choose $-$ sign so $t=0$ on initial line, $y=\sqrt{x}$

$t=y-\sqrt{2x-y^2}$

$s=\sqrt{2x-y^2}$

This may be true, but its getting a bit complicated. The χ-equations ①&② can be written $\frac{dy/dt}{dx/dt}=\frac{dy}{dx}=\frac{1}{y} \Rightarrow y^2/2=x+C$. The χ are given by choices of C & those which pass through $(1,1)$ & $(4,2)$ have $c=-1/2$ & $+2$ & are parabolas also. Those which pass through the initial data line cross the x-axis between $x=-2$ & $x=-1/2$ & cover the shaded region, the region in which the solution is determined.

The solution will be given by $\frac{du}{dx}=\frac{du/dt}{dx/dt}=\frac{x}{y}=\pm\frac{x}{\sqrt{2}\sqrt{x+C}}$ & integrating along the χ (C const)

this gives $u+B=\sqrt{2}\left[\frac{(x+c)^{3/2}}{3}-c(x+c)^{1/2}\right]$, but $(x+c)^{1/2}=y/\sqrt{2}$ so this is $\sqrt{2}\left[\frac{y^3}{2\sqrt{2}\,3}-(y^2/2-x)y\right]$

$=xy-y^3/3$. On the data line, $x=s^2$, $y=s$, the this is $2/3\,s^3$ as required so $B=0$ &

$u=xy-y^3/3$

7) a) $3z_x-4z_y=x^2$. χ eqns are $dx/3=dy/-4=\frac{dz}{x^2}$ i.e. $\frac{dy}{dx}=-4/3 \Rightarrow 4x+3y=$ Const $=\eta$ say

Transforming to x & η, $\frac{\partial}{\partial x}\to\frac{\partial}{\partial x}+\frac{\partial}{\partial\eta}\frac{d\eta}{dx}=\frac{\partial}{\partial x}+4\frac{\partial}{\partial\eta}$, $\frac{\partial}{\partial y}\to\frac{\partial}{\partial\eta}\frac{\partial\eta}{\partial y}=3\frac{\partial}{\partial\eta}$ & $3(z_x+4z_\eta)-4(3z_\eta)=x^2$

$\Rightarrow z_x=x^2/3$, $z=x^3/9+f(\eta)=x^3/9+f(4x+3y)$

Alternatively, along a χ with η fixed $dx/3=dz/x^2 \Rightarrow \int dz=\int x^2/3 \Rightarrow z=\frac{1}{9}x^3+f(\eta)$

b) $z_x-z_y=\sin x+\cos y$, χ: $dx/1=\frac{-dy}{1} \Rightarrow x+y=\eta=$ const. $\partial/\partial x\to\frac{\partial}{\partial x}+\frac{\partial}{\partial\eta}\cdot 1$, $\partial/\partial y\to\partial/\partial\eta\cdot 1$

$\Rightarrow (z_x+z_\eta)-z_\eta=\sin x+\cos(\eta-x) \Rightarrow z=f(\eta)-\cos x-\sin(\eta-x)=f(x+y)-\cos x-\sin y$

c) $z_x+z_y=\cos(x-y)$, χ: $dx/1=dy/1 \Rightarrow x-y=\eta=$ const, $\partial/\partial x\to\partial/\partial x+\partial/\partial\eta\cdot 1$, $\partial/\partial y\to\partial/\partial\eta(-1)$

$\Rightarrow (z_x+z_\eta)-z_\eta=\cos\eta \Rightarrow z=f(\eta)+x\cos(\eta)=f(x-y)+x\cos(x-y)$ etc

8) a) $xz_x-yz_y=0$ χ: $dx/x=dy/-y \Rightarrow \ln x=-\ln y+C \Rightarrow \ln(xy)=$ Const, $xy=\eta$ say. Change from (x,y) to (x,η) $\partial/\partial x\to\partial/\partial x+\partial/\partial\eta\cdot\partial\eta/\partial x=\partial/\partial x+y\,\partial/\partial\eta$, $\partial/\partial y\to\partial/\partial\eta\cdot\partial\eta/\partial y=x\,\partial/\partial\eta$ &

$x(z_x+yz_\eta)-y\,xz_\eta=0 \Rightarrow z_x=0$, $z=f(\eta)=f(xy)$

d) $xz_x+yz_y=(x+y)z$: χ: $dx/x=dy/y \Rightarrow \ln x-\ln y=\ln(x/y)=$ Const $\Rightarrow y/x=\eta$ say, const on χ

Change from (x,y) to (x,η). $\partial/\partial x\to\partial/\partial x+\partial/\partial\eta(-y/x^2)$, $\partial/\partial y\to\partial/\partial\eta\cdot 1/x \Rightarrow x(z_x-y/x^2\,z_\eta)+y/x\,z_\eta$

$=(x+\eta x)z \Rightarrow z_x=(1+\eta)z \Rightarrow z=f(\eta)e^{(1+\eta)x}=f(y/x)e^{x+y}$.

Sheet 5

a) $xzz_x - yzz_y = x^2-y^2$, χ: $\frac{dx}{dt} = xz$, $\frac{dy}{dt} = -yz$, $\frac{dz}{dt} = x^2-y^2$. We note $x\frac{dx}{dt} + y\frac{dy}{dt} = x^2z - y^2z$ $= (x^2-y^2)z = z\frac{dz}{dt}$ & integrating $x^2/2 + y^2/2 = z^2/2 + C \Rightarrow \underline{x^2+y^2-z^2 = \text{Const}}$. Also $y\frac{dx}{dt} + x\frac{dy}{dt} = yxz - xyz = 0 \Rightarrow \frac{1}{x}\frac{dx}{dt} + \frac{1}{y}\frac{dy}{dt} = 0 \Rightarrow \ln x + \ln y = \text{Const}$, $xy = \text{const}$

Lagranges method has solution $C_1 = f(C_2)$ ie $\underline{x^2+y^2-z^2 = f(xy)}$

b) $(x+z)z_x + yz_y = z+y^2$, χ: $\frac{dx}{dt} = x+z$, $\frac{dy}{dt} = y$, $\frac{dz}{dt} = z+y^2$. This looks tricky to find constants in the method of 1a. However $dy/dt = y \Rightarrow y = Ae^t$ ① with t taking us along a χ, a curve lying in the solution surface. Hence $dz/dt - z = y^2 = A^2e^{2t} \Rightarrow \frac{d}{dt}\left[ze^{-t}\right] = A^2e^t$ & $z = A^2e^{2t} + Be^t$ ②. Eliminating t between ①&② gives $z - y^2 = \frac{B}{A}y \Rightarrow \underline{z/y - y = \text{Const on } \chi}$

Continuing & finding x: $\frac{dx}{dt} - x = z = A^2e^{2t} + Be^t \Rightarrow \frac{d}{dt}\left[xe^{-t}\right] = A^2e^t + B \Rightarrow x = Ce^t + A^2e^{2t} + Bte^t$

Eliminating t, A & B gives $x = \frac{C}{A}y + y^2 + (z-y^2)\ln(y/A)$. Now we could rearrange this for C, another constant, but the choice of A is undetermined. Looking at the solution ① we could write this as $y = e^{t+\ln A}$ so that the choice of A corresponds only to the origin in t & we only really have two constants, B & C. Wlog we can choose $A=1$ so $y = e^t$, $B = \frac{z-}{y}$

& $C = [(x-y^2) - (z-y^2)\ln y]/y$ & the general solution is $f(C, B) = 0$.

c) $2x^2zz_x + \cos^2 y\, z_y = -xz^2$, χ: $\frac{dx}{dt} = 2x^2z$ ①, $\frac{dy}{dt} = \cos^2 y$ ②, $\frac{dz}{dt} = -xz^2$ ③. We note $z\frac{dx}{dt} + 2x\frac{dz}{dt} = 2x^2z^2 - 2x^2z^2 = 0$ so $\frac{1}{x}\frac{dx}{dt} + \frac{2}{z}\frac{dz}{dt} = 0$ & integrating $\ln x + 2\ln z = \text{Const} \Rightarrow$ $\underline{xz^2 = \text{Const} = C}$. Hence, from ② & ③ $\sec^2 y\frac{dy}{dt} = 1$ & $\frac{dz}{dt} = -C$ or $\sec^2 y\frac{dy}{dz} = -\frac{1}{C} \Rightarrow \tan y = -\frac{z}{C} + ($ for a second constant $B = \tan y + z/C = \tan y + z/xz^2 = \tan y + 1/xz$ & soln is $F\left(xz^2, \tan y + \frac{1}{xz}\right) = ($

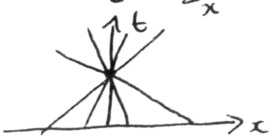
a) $z_x + x^2z_y = 0$, χ: $\frac{dx}{1} = \frac{dy}{x^2} \Rightarrow x^2dx - dy = 0 \Rightarrow x^3/3 - y = \text{Const}$, $x^3 - 3y = \eta$ say. Switch to (x, η) or alternatively note that ~~the third~~ another χ eqn is $dz/0 = \frac{dx}{1} \Rightarrow z = \text{const on a } \chi$ & $z = f(\eta) = f(x^3-3y)$. If $z = \cos y$ on $y = x^3$ then $\cos y = f(y-3y) = f(-2y)$ so $f(r) = \cos(-r/2)$ & $z = f(\eta) = \cos((3y - x^2)/2)$

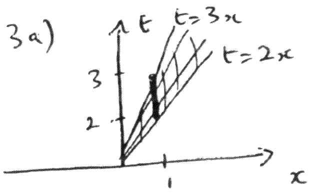
b) $z_x + \cos^2 y\, z_y = z$, χ: $\frac{dx}{1} = \frac{dy}{\cos^2 y} = \frac{dz}{z} \Rightarrow \sec^2 y\frac{dy}{dx} = 1 \Rightarrow \tan y = x + \text{Cst}$ & $C = x - \tan y$ is a constant. Also $\frac{dx}{dz} = \frac{1}{z} \Rightarrow x + \text{Cst} = \ln z \Rightarrow ze^{-x} = B$ another constant. The solution is $ze^{-x} = f(x - \tan y)$ using Lagrange's method. We have $z = \sec^2 y$ on $x = 0 \Rightarrow \sec^2 y\, e^{-0} = f(0 - \tan y)$ & so $f(r) = 1 + r^2$ as $1 + \tan^2 y = \sec^2 y$ & $ze^{-x} = 1 + (x - \tan y)^2$, $\underline{z = e^x + e^x(x - \tan y)^2}$

c) $z_x + 2e^{x-y}z_y = xe^x$, χ: $\frac{dx}{1} = \frac{dy}{2e^xe^{-y}} = \frac{dz}{xe^x}$. So $\frac{dy}{dx} = 2e^xe^{-y} \Rightarrow e^y dy = 2e^x dx$ & $e^y = 2e^x + C$ & one constant is $\underline{2e^x - e^y}$. Also $xe^x dx = dz$ & integrating $xe^x - e^x = z + \text{Const}$ & $\underline{z + e^x - xe^x}$ is another constant. The solution is $z + e^x - xe^x = f(2e^x - e^y)$. We know $z = xe^x$ on $y = 0$ so $\cancel{xe^x} + e^x - \cancel{xe^x} = f(2e^x - 1)$. Putting $r = 2e^x - 1$ we find $e^x = (r+1)/2$ & $f(r) = \frac{1}{2}(r+1)$ so that $\underline{z = xe^x - e^x + \frac{1}{2}(2e^x - e^y + 1)}$

d) $xz_x + (t-1)z_t = 0 \Rightarrow dx/x = dt/(t-1) = dz/0 \Rightarrow z = \text{const on lines } \ln x = \ln(t-1) + \text{Const}$

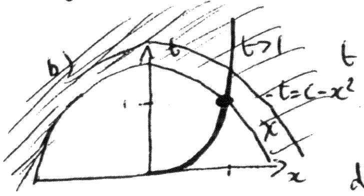
ie $\frac{t-1}{x} = C$, $t = 1 + Cx$. $z = f(C)$ as C marks out a χ, $z = f\left(\frac{t-1}{x}\right)$. If $z = F(x)$ on $t = 0$ then $F(x) = f(-1/x)$ & $f(r) = F(-1/r)$ & $\underline{z = f\left(x/(1-t)\right)}$ & the solution is singular, with all χ traces crossing at $t = 1$



3a) 

$x z_x + t z_t = (x+t)z$; χ: $\frac{dx}{x} = \frac{dt}{t} = \frac{dz}{(x+t)z}$. $\frac{dx}{x} = \frac{dt}{t} \Rightarrow \ln x =$
$\Rightarrow t/x =$ Const & χ are lines $t = Cx$ with those leaving $2 \le c \le 3$ passing through the boundary data so the solution's known in the region shown

We have on a χ $(t = Cx)$ $\frac{dx}{x} = \frac{dz}{(x+t)z} = \frac{dz}{(x+Cx)z} \Rightarrow dx = \frac{dz}{(1+c)z} \Rightarrow (1+c)x + f(c) = \ln z$
$\Rightarrow z = A(c)e^{(1+c)x} = A(t/x)e^{(t+x)}$. To find A we know $z=t$ on $x=1$ so $t = A(t/1)e^{t+1}$
so $A(t) = te^{-(1+t)}$ & $\underline{z = t/x\, e^{-(1+\frac{t}{x})} e^{(t+x)}}$

b) 

$t z_x - 2xt z_t = 2xz$; χ: $\frac{dx}{t} = \frac{dt}{-2xt} = \frac{dz}{2xz}$: $\frac{dx}{\cancel{t}} = \frac{dt}{-2x\cancel{t}} \Rightarrow -x^2 = t +$ Const
$\Rightarrow t + x^2 = c$. So $t = c - x^2$ ie inverted parabolas, gives the χ-traces in the (x,t) plane. The χ passing through $(1,1)$ has $c = 2$ & so solution's determined in the shaded region $t > 2 - x^2$. On these χ, $\frac{dx}{t} = \frac{dz}{2xz}$
& using $t = c - x^2$ $\frac{2x}{c-x^2}dx = \frac{dz}{z} \Rightarrow -\ln(c-x^2) = \ln z + \underset{=f(c)}{\text{Const}} \Rightarrow z = \frac{A(c)}{c-x^2}$
$= A(t+x^2)/[t+x^2-x^2] = \frac{A(t+x^2)}{t}$ for some function A. Now $z=t$ on $t = x^2$ so
$t = \frac{A(t+t)}{t} \Rightarrow A(2t) = t^2$, $A(r) = (r/2)^2$ & $\underline{z = \frac{1}{4t}(t+x^2)^2}$

4) a) $z z_x + y z_y = z^2$; χ: $\frac{dx}{z} = \frac{dy}{y} = \frac{dz}{z^2}$. So we see $\frac{dx}{\cancel{z}} = \frac{dz}{z^{\cancel{2}}} \Rightarrow dx = \frac{dz}{z}$, $x + C = \ln z$
$z = Ce^x \Rightarrow \underline{ze^{-x}}$ is Constant, C_1 say. Also $\frac{dy}{y} = \frac{dz}{z^2} \Rightarrow \ln y = -\frac{1}{z} +$ Const, $y = Ce^{-1/z}$ &
$C_2 = ye^{1/z}$ is a second constant. The general solution, using Lagrange's method is
$C_2 = f(C_1) \Rightarrow ye^{1/z} = f(ze^{-x})$. On $z = 1$, $\ln y = e^x$ so $ye^{1/1} = f(1/\ln y)$ & $f(r) =$
$ee^{1/r}$ & $ye^{1/z} = e^{\cancel{1}} e^{\frac{1}{ze^{-x}}}$ or $\ln y + 1/z = 1 + \frac{1}{ze^{-x}} \Rightarrow \frac{1}{z}\left(1 - \frac{1}{e^{-x}}\right) = 1 - \ln y$, $\underline{z = \frac{e^x - 1}{\ln y - 1}}$

b) $z z_x + xy^2 z_y = xz^3$; χ: $\frac{dx}{z} = \frac{dy}{xy^2} = \frac{dz}{xz^3}$. As above, $\frac{dx}{\cancel{z}} = \frac{dz}{xz^{\cancel{3}2}} \Rightarrow x\,dx = dz/z^2$
$\Rightarrow \frac{x^2}{2} + \frac{1}{z} = C_1$. Also $\frac{dy}{\cancel{x}y^2} = \frac{dz}{\cancel{x}z^3} \Rightarrow -\frac{1}{y} = -\frac{1}{2z^2} + C_2$, $C_2 = \frac{1}{2z^2} - \frac{1}{y}$. Lagrange's method gives the solution as $C_1 = f(C_2)$ ie $\frac{x^2}{2} + \frac{1}{z} = f\left(\frac{1}{2z^2} - \frac{1}{y}\right)^{*}$. To find f use $x = 0, y = 1$ so that $0 + \frac{1}{z} = f\left(\frac{1}{2z^2} - \frac{1}{1}\right)$, put $r = \frac{1}{2z^2} - 1 \Rightarrow \frac{1}{z} = \sqrt{2(r+1)}$ so $f(r) = \sqrt{2(r+1)}$ & using this & squaring * $x^4/4 + \frac{x^2}{z} + \frac{\cancel{1}}{\cancel{z^2}} = 2\left(\frac{\cancel{1}}{\cancel{2z^2}} - \frac{1}{y} + 1\right) \Rightarrow \frac{1}{z} = 2/x^2 - x^2/4 - 2/x^2y$

5) a) $(x+u)u_x + (y+u)u_y = 0$; χ: $dx/(x+u) = dy/(y+u) = du/0$, Use Lagrange's method; we have $\frac{du}{dt} = 0 \Rightarrow u =$ Const $= C_1$, $\frac{dx}{dt} - \frac{dy}{dt} = (x+\cancel{u}) - (y+\cancel{u}) = x - y \Rightarrow \frac{d(x-y)}{(x-y)} = dt$.
Actually is best not to introduce t here; we have also $dy/dx = \frac{y+u}{x+u}$, but we know $u = C_1$ so we have $\frac{dy}{dx} = \frac{y+C_1}{x+C_1}$ which we can integrate: $\frac{dy}{y+C_1} = \frac{dx}{x+C_1} \Rightarrow \ln(y+C_1) = \ln(x+C_1) +$ Const & now replace C_1 & find $(y+u)/(x+u) = C_2$. General solution is $C_1 = f(C_2)$, $u = f\left(\frac{y+u}{x+u}\right)$

b) $(x^2+3y^2+3u^2)u_x - 2xyu_y = -2xu$; χ: $dx/(x^2+3y^2+3u^2) = dy/-2xy = \frac{du}{-2xu}$. The last equation is $\frac{dy}{y} = \frac{du}{u} \Rightarrow \ln y = \ln u + C \Rightarrow u/y = C_1$. We can use this and $dx/(x^2+3y^2+3u^2) = \frac{du}{-2xu}$, writing, $y = u/C_1$ to give $\frac{du}{dx} = \frac{-2xu}{x^2+3u^2(1+1/C_1^2)}$. This could be integrated, but let's try another way. $\frac{dx}{dt} = x^2+3y^2+3u^2$, $\frac{dy}{dt} = -2xy$, $\frac{du}{dt} = -2xu$, so $x\frac{dx}{dt} + y\frac{dy}{dt} + z\frac{dz}{dt} = x^3 + 3y^2x + 3u^2x$
$-2xy^2 - 2xu^2 = x(x^2+y^2+u^2) = (x^2+y^2+u^2)\left(\frac{-1}{2y}\frac{dy}{dt}\right)$ using $\frac{dy}{dt} = -2xy$. Rearranging & noting $x\frac{dx}{dt} + y\frac{dy}{dt} + z\frac{dz}{dt} = \frac{1}{2}\frac{d}{dt}(x^2+y^2+z^2)$ gives $\frac{\frac{d}{dt}(x^2+y^2+z^2)}{(x^2+y^2+z^2)} + \frac{1}{y}\frac{dy}{dt} = 0$
$\Rightarrow y(x^2+y^2+z^2) = C_2$. Solution is $C_2 = f(C_1)$, $y(x^2+y^2+z^2) = f(u/y)$

6) $u_t + u^2 u_x = 0$, $u(x,0) = x$. χ are $\frac{dt}{1} = \frac{dx}{u^2} = \frac{du}{0}$ So u = constant, C on χ &
the χ eqns have $\frac{dt}{dx} = \frac{1}{u^2} = \frac{1}{C^2}$ so $t = \frac{x}{C^2} + f(C)$. Recall $u = C$ & so $t = \frac{x}{u^2} + f(u)$.
To find f use $u = x$ on $t=0$ so $0 = x/x^2 + f(x) \Rightarrow f(x) = -1/u$ & $t = \frac{x}{u^2} - \frac{1}{u}$
Solve this for u, $u^2 t + u - x = 0$, $u = (-1 \pm \sqrt{1+4xt})/2t$. Choose + root so that limiting soln as $t \to 0$ is correct, $\underline{u = (\sqrt{1+4xt} - 1)/2t}$

7) a) $(y-u)u_x + (u-x)u_y = x-y$: $dx/(y-u) = dy/(u-x) = du/(x-y) = dt$, say. Hence
$\frac{dx}{dt} + \frac{dy}{dt} + \frac{du}{dt} = y-u+u-x+x-y = 0 = \frac{d}{dt}(x+y+u) \Rightarrow x+y+u = C_1$ constant on χ. Also
$x\frac{dx}{dt} + y\frac{dy}{dt} + u\frac{du}{dt} = x(y-u)+y(u-x)+u(x-y) = 0 = \frac{1}{2}\frac{d}{dt}(x^2+y^2+z^2) \Rightarrow x^2+y^2+z^2 = C_2$. Lagranges method
gives $x+y+u = f(x^2+y^2+u^2)$ & if $u=0$ when $y=2x$ then $x+2x+0 = f(x^2+4x^2+0) \Rightarrow f(5x^2) = 3x$
If $r = 5x^2$ then $x = \pm\sqrt{r/5}$ & $(x+y+u) = 3(\pm 1)(x^2+y^2+u^2)^{1/2}/\sqrt{5} \Rightarrow \underline{5(x+y+u)^2 = 9(x^2+y^2+u^2)}$

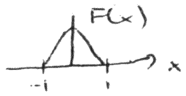
b) $(2xy+2y^2+u)u_x - (2x^2+2xy+u)u_y = 2u(x-y)$: $\frac{dx}{dt} = 2xy+2y^2+u$, $\frac{dy}{dt} = -2x^2-2xy-u$
$\frac{du}{dt} = 2u(x-y)$. We see $x\frac{dx}{dt} + y\frac{dy}{dt} = \cancel{2x^2y} + \cancel{2xy^2} + ux - \cancel{2x^2y} - \cancel{2xy^2} - uy = \frac{1}{2}\frac{du}{dt}$
$\Rightarrow \frac{d}{dt}\left(\frac{1}{2}x^2 + \frac{1}{2}y^2 - \frac{1}{2}u\right) = 0$ & $x^2+y^2-u = C_1$ Also $\frac{dx}{dt} + \frac{dy}{dt} = (2y^2-2x^2) = 2(y-x)(y+x)$
$\Rightarrow \frac{d}{dt}(x+y)/x+y = 2(y-x) = -\frac{du/dt}{u} \Rightarrow \frac{d}{dt}\left[\ln(x+y) + \ln u\right] = 0$ & so $(x+y)u = C_2$
Lagranges method gives $(x+y)u = f(x^2+y^2-u)$, $u = 2x^2$ on $y=0$ gives $(x+0)2x^2 =$
$f(x^2+0-2x^2) = f(-x^2)$. If $r = -x^2$, $x = \pm\sqrt{-r}$ & $2x^3 = \pm 2(-r)^{3/2} = f(r)$ &
$(x+y)u = \pm 2(u-x^2-y^2)^{3/2}$ or $\underline{u^2(x+y)^2 = 4(u-x^2-y^2)^3}$.

c) $(3y-2u)u_x + (u-3x)u_y = 2x-y$: $\frac{dx}{dt} = 3y-2u$, $\frac{dy}{dt} = u-3x$, $\frac{du}{dt} = 2x-y$. Hence
$x\,dx/dt + y\,dy/dt + u\,du/dt = \cancel{3xy} - \cancel{2xu} + \cancel{uy} - \cancel{3xy} + \cancel{2xu} - \cancel{uy} = 0 \Rightarrow \frac{1}{2}\frac{d}{dt}(x^2+y^2+u^2) = 0$
& $x^2+y^2+u^2 = C_1$. Also $\frac{dx}{dt} + 2\frac{dy}{dt} + 3\frac{du}{dt} = \cancel{3y} - \cancel{2u} + \cancel{2u} - \cancel{6x} + \cancel{6x} - \cancel{3y} = 0 \Rightarrow \frac{d}{dt}(x+2y+3u) = 0$
& $x+2y+3u = C_2$. Lagrange's method gives $C_1 = f(C_2)$, $x^2+y^2+u^2 = f(x+2y+3u)$. If
$u=0$ when $x=y$ then $2x^2 = f(3x)$ & $f(r) = 2(r/3)^2$ & $\underline{9(x^2+y^2+u^2) = 2(x+2y+3u)^2}$

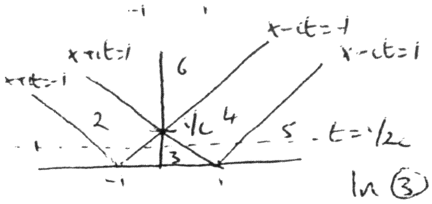
8) a) $xuu_x + y^2u_y = u^2$, $u = s$ on $x = s^{-1}$, $y = 2s$, has χ eqns $\frac{dx}{dt} \overset{①}{=} xu$, $\frac{dy}{dt} \overset{②}{=} y^2$, $\frac{du}{dt} \overset{③}{=} u^2$, $u = s$, $y = 2s$, $x = 1/s$ on $t=0$
② $\Rightarrow -\frac{1}{y} = t + \text{Const}$ & using $y = 2s$ on $t=0$, $t = \frac{1}{2s} - \frac{1}{y}$. Similarly $t = \frac{1}{s} - \frac{1}{u}$ from ③. Finally
① gives $\frac{1}{x}\frac{dx}{dt} = u = \left(\frac{1}{s} - t\right)^{-1} = \frac{s}{1-st} \Rightarrow \int \frac{1}{x}\,dx = \int \frac{s}{1-st}\,dt \Rightarrow \ln x = -\ln(1-st) + \text{const}$
$\Rightarrow x(1-st) = C = 1/s$ using the bc $\Rightarrow x = \frac{1}{s(1-st)}$ & $u = \frac{s}{1-st}$, $y = \frac{2s}{1-2st}$. Eliminating
s&t we see $ux = \frac{s}{1-st} \cdot \frac{1}{s(1-st)} = \frac{1}{(1-st)^2} \Rightarrow 1-st = \frac{1}{\sqrt{ux}} \Rightarrow st = 1 - \frac{1}{\sqrt{ux}}$ & from
$s\tau = \frac{1}{1-st} = \sqrt{ux}$, $s = \frac{\sqrt{u}}{\sqrt{x}}$, so $y = \frac{2\sqrt{u}}{\sqrt{x}} \cdot \frac{1}{1-2\left(1-\frac{1}{\sqrt{ux}}\right)} = \underline{\frac{2u}{2+\sqrt{xu}}}$.

b) $x^2u_x + uu_y = 1$, $u = 0$ on $x = s$, $y = 1-s$, has χ eqns $\frac{dx}{dt} \overset{①}{=} x^2$, $\frac{dy}{dt} \overset{②}{=} u$, $\frac{du}{dt} \overset{③}{=} 1$, $u = 0$, $x = s$, $y = 1-s$ on $t=0$
③ $\Rightarrow u = t$ using bc $\Rightarrow dy/dt = u = t$, $y = \frac{1}{2}t^2 + (1-s)$ using bc, $dx/dt = x^2 \Rightarrow -\frac{1}{x} = t + C$
$= t - \frac{1}{s}$ using bc so $\underset{(\alpha)}{u = t}$, $\underset{(\beta)}{y = \frac{1}{2}t^2 + (1-s)}$, $x = \frac{s}{1-st}$ (γ) α&γ $\Rightarrow (1-st)x = s$
$\Rightarrow s = \frac{x}{1+ux}$ & $\underline{y = u^2/2 + 1 - \frac{x}{1+ux}}$

2401 Sheet 7

1) 

D'Alembert's solution gives $z(x,t)=\frac{1}{2}\big(\underset{A}{F(x-ct)}+\underset{B}{F(x+ct)}\big)$



In ① $x+ct$ & $x-ct<-1$, $A=B=0$ $\underline{Z=0}$

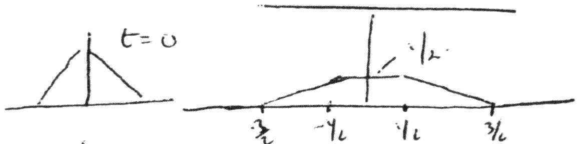
In ② $-1<x+ct<1$, $x-ct<-1$ $B=1-|x+ct|$, $\underline{Z=\frac{1}{2}(1-|x+ct|)}$

In ③ $-1<x-ct<1$ & $-1<x+ct<1$, $A=1-|x-ct|$, $B=1-|x+ct|$ & $\underline{z=1-\frac{1}{2}(|x-ct|+|x+ct|)}$

In ④ $x+ct>1$ & $-1<x-ct<1$, $B=0$, $A=1-|x-ct|$, $\underline{Z=\frac{1}{2}(1-|x-ct|)}$.

In ⑤ $x+ct>1$, $x-ct>1$, $A=B=0$, $\underline{z=0}$.

In ⑥ $x-ct<-1$, $x+ct>1$, $A=B=0$, $\underline{z=0}$.



If $t=1/2c$, the line $t=1/2c$ intersects these x running from -1 & $+1$ at $-3/2, -1/2, 1/2, 3/2$

So if $x<-3/2$, region 1, $\underline{z=0}$. If $-3/2<x<-1/2$, region 2, $z=\frac{1}{2}(1-|x+ct|)$ $=\frac{1}{2}(1-|x+\frac{1}{2}|)=\frac{1}{2}(1-(-1)(x+\frac{1}{2}))$ (as $x+\frac{1}{2}<0$) $=\underline{\frac{1}{2}(3/2+x)}$. If $-1/2<x<1/2$, region 3, $z=1-\frac{1}{2}(|x-\frac{1}{2}|+|x+\frac{1}{2}|)=1-\frac{1}{2}(-(x-\frac{1}{2})+(x+\frac{1}{2}))=\underline{\underline{1/2}}$.

If $1/2<x<3/2$, region 4, $z=\frac{1}{2}(1-|x-\frac{1}{2}|)=\frac{1}{2}(1-(x-\frac{1}{2}))=\frac{1}{2}(3/2-x)$. If $x>3/2$, region 5, $z=0$

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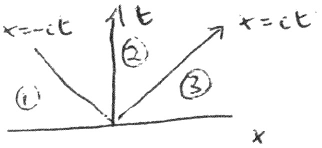
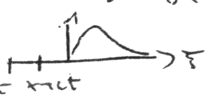
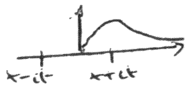
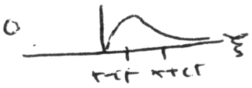
2) 

D'Alembert's solution gives $z(x,t)=\frac{1}{2c}\int_{x-ct}^{x+ct}G(\xi)\,d\xi$

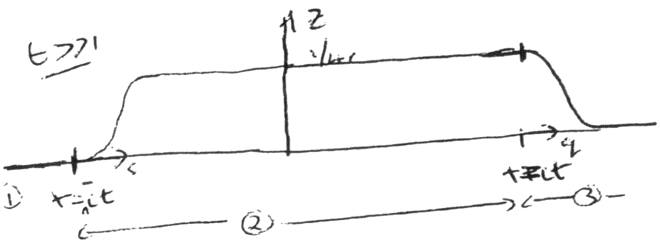


In ① $x-ct<0$, $x+ct<0$  , $z=\frac{1}{2c}\int_{x-ct}^{x+ct}0\,d\xi=\underline{0}$.

In ② $x-ct<0$, $x+ct>0$  $z=\frac{1}{2c}\left\{\int_{x-ct}^{0}0\,d\xi+\int_0^{x+ct}\xi e^{-\xi^2}d\xi\right\}$

$=\frac{1}{2c}\left[-\frac{1}{2}e^{-\xi^2}\right]_0^{x+ct}=\underline{\frac{1}{4c}\left[1-e^{-(x+ct)^2}\right]}$. In ③ $x-ct>0$, $x+ct>0$ 

$z=\frac{1}{2c}\int_{x-ct}^{x+ct}\xi e^{-\xi^2}d\xi=\frac{1}{2c}\left[-\frac{1}{2}e^{-\xi^2}\right]_{x-ct}^{x+ct}=\frac{1}{4c}\left(e^{-(x-ct)^2}-e^{-(x+ct)^2}\right)$



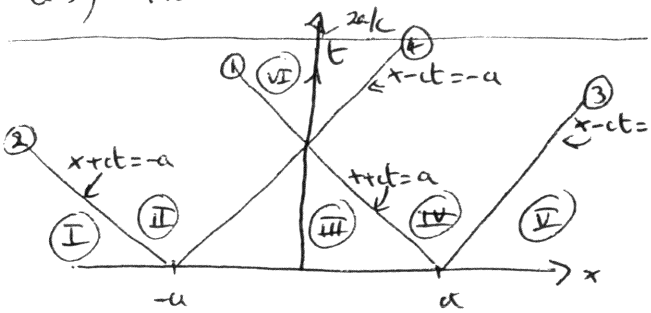
① $x-ct<0$, $z=0$

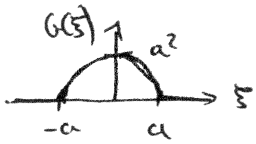
② If $x+ct$ is large $e^{-(x+ct)^2}\to 0$ & $z\to 1/4c$. However $x+ct$ is not large near $x=-ct$. Near here, put $x=-ct+s$, $z=\frac{1}{4c}(1-e^{-s^2})$   & solution is as shown.

③ $x+ct$ is large & $z\approx\frac{1}{4c}\left(e^{-(x-ct)^2}-0\right)=\frac{1}{4c}e^{-q^2}$, $q=x-ct$ & solution is as shown

Sheet 6

Q3) From D'Alembert's solution $z(x,t)=\frac{1}{2c}\int_{x-ct}^{x+ct} G(\xi)\,d\xi$ $\quad G(\xi)=\begin{cases}a^2-\xi^2 & |\xi|<a\\ 0 & |\xi|>a\end{cases}$

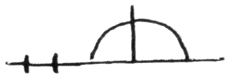




$|x+ct|=a \Rightarrow x+ct=a$, $t=\frac{1}{c}(a-x)$ line ①
or $-(x+ct)=a$, $t=-\frac{1}{c}(a+x)$ line ②

$|x-ct|=a \Rightarrow x-ct=a$, $t=-\frac{1}{c}(a-x)$ line ③
or $-(x-ct)=a$, $t=\frac{1}{c}(a+x)$ line ④

In I $\quad x-ct<a$ & $x+ct<a$, $\quad z=\frac{1}{2c}\int_{x-ct}^{x+ct}0\,d\xi=0$



In II $\quad x+ct>-a$, but $<a$, $\quad x-ct<-a$

$$z=\frac{1}{2c}\left[\int_{x-ct}^{-a}G(\xi)d\xi+\int_{-a}^{x+ct}G(\xi)d\xi\right]$$

$$=\frac{1}{2c}\left[\int_{x-ct}^{-a}0\,d\xi+\int_{-a}^{x+ct}(a^2-\xi^2)d\xi\right]=\frac{1}{2c}\left[a^2\xi-\frac{\xi^3}{3}\right]_{-a}^{x+ct}$$

In III $\quad x+ct>-a$ but $<a$ $\quad x-ct>-a$ but $<a$

$$z=\frac{1}{2c}\int_{x-ct}^{x+ct}a^2-\xi^2\,d\xi$$

$$=\frac{1}{2c}\left[a^2\xi-\xi^3/3\right]_{x-ct}^{x+ct}$$

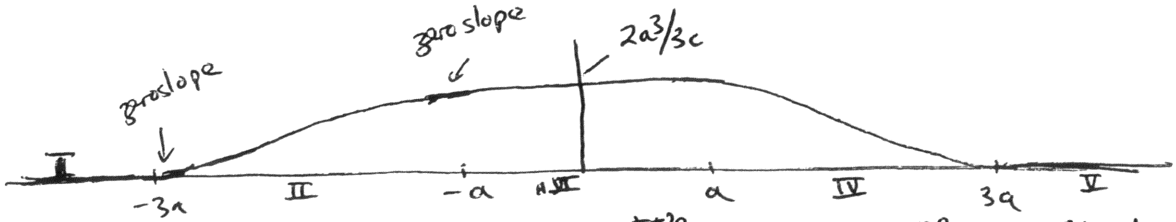
Now being briefer

In IV $\quad x+ct>a$, $x-ct>-a$ but $<a$ $\quad z=\frac{1}{2c}\left[a^2\xi-\xi^3/3\right]_{x-ct}^{a}$

In V $\quad x+ct>a$ $x-ct>a$ $\quad z=0$

In VI $\quad x+ct>a$, $x-ct<-a$ $\quad z=\frac{1}{2c}\left[a^2\xi-\xi^3/3\right]_{-a}^{a}$

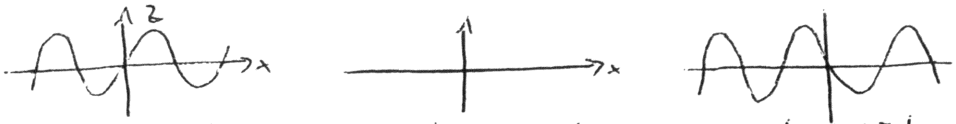
If $t=2a/c$. Lines ① & ④ meet at $x=0$ if $t=a/c$. So if $t=2a/c$ we are in region I for $-\infty<x<-3a$ when $z=0$, region II for $-3a<x<-a$ when $z=\frac{1}{2c}\left[a^2\xi-\xi^3/3\right]_{-a}^{x+2a}$ ($x+ct=x+c(2a/c)=x+2a$), region III for $-a<x<a$ when $z=\frac{1}{2c}\left[a^2\xi-\xi^3/3\right]_{-a}^{a}=2\frac{1}{2c}\left[a^2a-a^3/3\right]=2a^3/3c$, region IV for $a<x<3a$ when $z=\frac{1}{2c}\left[a\xi-\xi^3/3\right]_{x-2a}^{a}$ & region V for $x>3a$ when $z=0$



Note that in II $z=\frac{1}{2c}\left[a^2\xi-\xi^3/3\right]_{-a}^{x+2a}=\frac{1}{2c}\left[a^2\xi-\xi^3/3\right]_{-a}^{a}=2a^3/3c$ at $x=-a$ and $z=\frac{1}{2c}\left[a^2\xi-\xi^3/3\right]_{-a}^{-a}=0$ at $x=-3a$. Also $\frac{\partial z}{\partial x}=\left[a^2-\xi^2\right]^{x+2a}=0$ at $x=-3a$ or $x=-a$ (when $x+2a=-a$ or a).

4) i) $z = \frac{1}{2}[F(x+ct)+F(x-ct)]$, $F(x)=\sin\pi x$

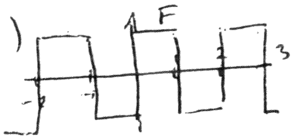
$$z(x,t) = \frac{1}{2}\left(\sin\pi(x+ct)+\sin\pi(x-ct)\right) = \sin\pi x\cos\pi c t .$$



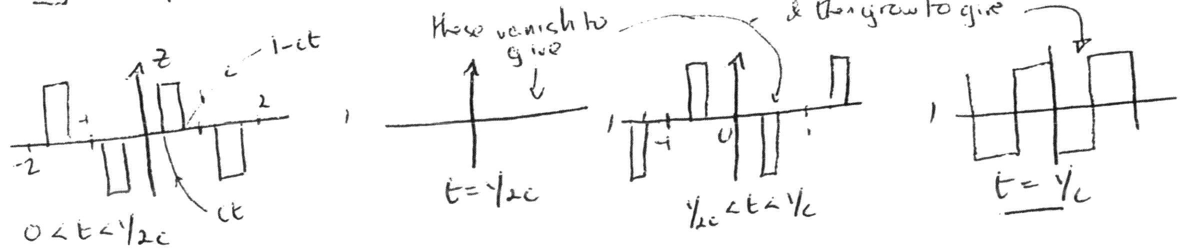
$t=0$, $\cos\pi ct=1$ $\quad$ $t=1/2c$, $\cos\pi ct=0$, $\quad$ $t=1/c$ $\cos\pi ct=-1$ $\quad$ & so on

The leftward & rightward travelling components interfere to create a standing wave

ii)



$$z(x,t) = \frac{1}{2}\begin{bmatrix}1 & 2n\le x+ct<2n+1\\ -1 & 2n+1\le x+ct<2n+2\end{bmatrix} + \frac{1}{2}\begin{bmatrix}1 & 2n<x-ct<2n+1\\ -1 & 2n+1<x-ct<2n\ldots\end{bmatrix}$$



5) $z_{1tt}=c_1^2z_{1xx}$ $\quad m \quad$ $z_{2tt}=c_2^2z_{2xx}$

$\rho_1 \quad 0 \quad \rho_2$

incident $\quad$ reflected $\quad$ transmitted

At $x=0$ ① z is cts

but $m z_{1tt} = m z_{2tt} = T(z_{2x}-z_{1x})$ ②

for $x<0$, write $z_1 = Ae^{i(\omega t-kx)} + Re^{i(\omega_r t+k_r x)}$ (incident, reflected) $\quad$ (allow for the present ω_r & k_r, ω_s & k_s different, even though they may well be alike)

$x>0$ write $z_2 = Se^{i(\omega_s t-k_s x)}$

① z cts at $x=0$ $\Rightarrow$ $Ae^{i\omega t}+Re^{i\omega_r t} = Se^{i\omega_s t}$ $\Rightarrow$ $\omega=\omega_r=\omega_s$ (as true for all t) & $A+R=S$

Now $\omega/k=c_1$ & $\omega_r/k_r=c_1$ & $\omega_s/k_s=c_2$, so we conclude $k_r=k=\omega/c_1$, $k_s=\omega/c_2$

② $\Rightarrow$ $M(-\omega_s^2)S = T\left(-ik_sS-(-Aik+iRk_r)\right)$, the $e^{i\omega t}$ cancelling away

$\Rightarrow -M\omega^2S = ikT\left(A-R-\frac{k_s}{k}S\right)$, but $R=S-A$ so

$M\omega^2S = ikT\left(\frac{k+k_s}{k}S-2A\right)$ $\Rightarrow$ $S=\frac{2k}{k+k_s+ipk}A$, $R=\frac{k-k_s-ipk}{k+k_s+ipk}A$

& $p=\frac{\omega^2M}{Tk}$, $k_s=\frac{c_1}{c_2}k=\sqrt{\rho_2/\rho_1}\,k$

If $M\to\infty$, $p\to\infty$ & $S\to0$, $R\to-A$ (perfect reflection)

If $M=0$ $\rho_1=\rho_2$ $p=0$, $S=A$, $R=0$, (perfect transmission)

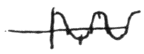
If $M=0$ $\rho_1\ll\rho_2$ ie $p=0$ but $k_s\gg k$, $S\to0$ & $R\to-1$, perfect reflection from heavy string 2.

Q6 $c=\sqrt{T/\rho}$. Look for solutions of $z_{tt}=c^2 z_{xx}$ of the form $z=X(x)T(t)$ with $X(\pm L)=0$, giving $T''/c^2T = X''/X$ = Constant. To satisfy the homogenous boundary conditions the constant needs to be negative (so we get oscillatory solutions) Call it $-p^2$ so that $X''+p^2X=0$ & $T''+p^2c^2T=0$. The frequencies are the values of cp. $X(x)=A\cos px + B\sin px$ & the boundary conditions require $0=A\cos pL+B\sin pL$ & $0=A\cos p(-L)+B\sin p(-L)=A\cos pL-B\sin pL$.

Write this as $\begin{pmatrix}\cos pL & \sin pL\\ \cos pL & -\sin pL\end{pmatrix}\begin{pmatrix}A\\B\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}$ & we get non zero solutions for A & B if the determinant of the matrix is zero, ie $-2\cos pL\sin pL=-\sin 2pL=0$

$\Rightarrow$ $2pL=n\pi$, $pL=n\pi/2$, $\cos pL=\cos n\pi/2=0$ if n is odd, $(-1)^{n/2}$ if n is even.

$\sin pL=\sin n\pi/2=0$ if n is even, $(-1)^{\frac{n-1}{2}}$ if n is odd



So the frequencies are $\frac{cn\pi}{2L}$, $n=1,2,3\ldots$. If n is odd then the matrix equation for A & B is $\begin{pmatrix}0 & (-1)^{\frac{n-1}{2}}\\ 0 & -(-1)^{\frac{n-1}{2}}\end{pmatrix}\begin{pmatrix}A\\B\end{pmatrix}=0 \Rightarrow B=0$ & $X=A\cos\frac{n\pi x}{2L}$ – even solutions in x

If n is even the system is $\begin{pmatrix}(-1)^{n/2} & 0\\ (-1)^{n/2} & 0\end{pmatrix}\begin{pmatrix}A\\B\end{pmatrix} \Rightarrow A=0$ & $X=B\sin\frac{n\pi x}{L}$ – odd solution in x

The general solution is
$$z=\sum_{n\ \text{even}}\sin\frac{n\pi x}{L}\left(C_n\sin\frac{n\pi ct}{L}+D_n\cos\frac{n\pi ct}{L}\right)+\sum_{n\ \text{odd}}\cos\frac{n\pi x}{L}\left(C_n\sin\frac{n\pi ct}{L}+D_n\cos\frac{n\pi ct}{L}\right)$$

At $t=0$ we have an initial condition on $\partial z/\partial t$ (so we will need the sin dependence on t) & the condition is even in x, so we will need the $\cos n\pi x/L$ with n odd.

$$z(x,t)=\sum_{n\ \text{odd}}\cos\frac{n\pi x}{L}C_n\sin\frac{n\pi ct}{2L}=\sum_{n=0}^{\infty}C_n\cos\frac{(2n+1)\pi x}{L}\sin\frac{(2n+1)\pi ct}{2L}$$

The solution at $x=0$, $z(0,t)=\sum_{n=0}^{\infty}C_n\sin\frac{(2n+1)\pi ct}{2L}$ & C_n are found from initial conditions so that

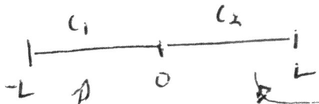
$$\frac{\partial z}{\partial t}=\sum_{n=0}^{\infty}C_n\cos\frac{(2n+1)\pi x}{L}\cdot\frac{(2n+1)\pi c}{2L}\overset{1}{\cancel{\cos\frac{(2n+1)\pi c\overset{t=0}{0}}{2L}}}=L^2-x^2$$

$\Rightarrow$, on multiplying by $\cos\frac{(2m+1)\pi x}{L}$ & integrating from $-L$ to L

$$C_m\cdot\frac{1}{2}\cdot 2L\cdot\frac{(2m+1)\pi c}{2L}=\int_{-L}^{L}(L^2-x^2)\cos\frac{(2m+1)\pi x}{L}dx=2\int_0^L(L^2-x^2)\cos\frac{(2m+1)\pi x}{L}dx$$

$$\Rightarrow C_m=\frac{64L^3}{\pi c^4}\frac{(-1)^n}{(2n+1)^4}$$ after integrating by parts

7)



$x<0$

$$z = Ae^{i\omega t}e^{-ikx} + Be^{i\omega t}e^{ikx}$$

$x>0$

$$z = Ce^{i\omega t}e^{-i\bar{k}x} + De^{i\omega t}e^{i\bar{k}x}$$

$c_1 = k\omega$, $k = \omega/c_1$, $\frac{\bar{k}}{k} = \frac{c_1}{c_2}$
$c_2 = \bar{k}\omega$, $\bar{k} = \omega/c_2$

Look for solutions with reflected & transmitted modes (incoming modes) in each half. Expect all to have same frequency

BC's are

$z = 0$ at $\pm L$, z cts at 0 & $\frac{\partial z}{\partial x}$ cts at 0

Hence

$Ae^{iLk} + Be^{-iLk} = 0$, $A + B = C + D$

$Ce^{-iL\bar{k}} + De^{iL\bar{k}} = 0$, $-ikA + ikB = -i\bar{k}C + i\bar{k}D$

This leads to the set of equations, written in matrix form

$$\begin{pmatrix} e^{iLk} & e^{-iLk} & 0 & 0 \\ 0 & 0 & e^{-i\bar{k}L} & e^{i\bar{k}L} \\ 1 & 1 & -1 & -1 \\ -k & k & \bar{k} & -\bar{k} \end{pmatrix}\begin{pmatrix} A \\ B \\ C \\ D \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

which has nonzero solutions if determinant is zero

$$\Rightarrow e^{iLk}\left(-1\left(-\bar{k}e^{-i\bar{k}L} - \bar{k}e^{i\bar{k}L}\right) + k\left(-e^{-i\bar{k}L} + e^{i\bar{k}L}\right)\right)$$

$$- e^{-iLk}\left(-\left(-\bar{k}e^{-i\bar{k}L} - \bar{k}e^{i\bar{k}L}\right) - k\left(-e^{-i\bar{k}L} + e^{i\bar{k}L}\right)\right) = 0$$

$$\Rightarrow e^{iLk}\left(2\bar{k}\cos\bar{k}L\right) + 2ki\sin\bar{k}L\Big) = e^{-iLk}\left(2\bar{k}\cos\bar{k}L - 2ki\sin\bar{k}L\right) = 0$$

$$\Rightarrow \bar{k}\cos\bar{k}L\left(e^{iLk} - e^{-iLk}\right) + ki\sin\bar{k}L\left(e^{iLk} + e^{-iLk}\right) = 0$$

$$\Rightarrow \bar{k}\cos\bar{k}L\sin Lk + k\sin\bar{k}L\cos Lk = 0$$

$\div$ through by $\cos Lk\cos\bar{k}L$ & write $k = \omega/c_1$, $\bar{k} = \omega/c_2$ & $\div$ by ω

$$\frac{1}{c_2}\tan\frac{L\omega}{c_1} + \frac{1}{c_1}\tan\left(\frac{L\omega}{c_2}\right) = 0 \quad \text{as required.}$$

An equation for frequency of oscillation, ω

8)  $z_x = 0$ $\qquad z_{tt} = c^2 z_{xx}$. Try $z = X(x)T(t) \Rightarrow T''X = c^2 \ldots$

$\Rightarrow X''/X = \dfrac{T''}{c^2 T} = \lambda$, a const. We have to impose $X(0) = X'(L) = 0$ ... need trigonometric functions in X. Hence $\lambda < 0$, $\lambda = -p^2$ say. ($\lambda = 0$ gives $X'' = 0$, $X = Ax + B$ & we cannot impose $X(0) = X'(L) = 0$, unless $A = B = 0$). $X'' + p^2 X = 0$ & $T'' + c^2 p^2 T = 0$. We have $X(x) = A\cos px + B\sin px$. $X(0) = 0 \Rightarrow A = 0$

$X'(L) = 0 \Rightarrow pL = (n + \tfrac{1}{2})\pi$. Hence $X(x) = \sin(n+\tfrac{1}{2})\pi x/L$. The solution for T is $T(t) = C\cos[(n+\tfrac{1}{2})\pi ct/L] + D\sin[(n+\tfrac{1}{2})\pi ct/L]$, but we require $T'(0) = 0$ as the string is released from rest. Hence $D = 0$. The soln is

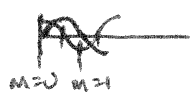
$$z(x,t) = \sum_{n=0}^{\infty} A_n \cos[(n+\tfrac{1}{2})\pi ct/L]\sin[(n+\tfrac{1}{2})\pi x/L]$$ with A_n chosen so that

$$z(x,0) = \sum_{n=0}^{\infty} A_n \cdot 1 \cdot \sin[(n+\tfrac{1}{2})\pi x/L] = \varepsilon x/L.$$ Multiplying by $\sin(m+\tfrac{1}{2})\dfrac{\pi x}{L}$

& integrating gives $$\sum_0^{\infty} A_n \int_0^L \sin(n+\tfrac{1}{2})\frac{\pi x}{L}\sin(m+\tfrac{1}{2})\frac{\pi x}{L}\,dx = \sum_0^{\infty} A_n \cdot \tfrac{1}{2}L\cdot\delta_{nm} = \frac{L}{2}A_m$$

$$\frac{\varepsilon}{L}\int_0^L x\sin(m+\tfrac{1}{2})\frac{\pi x}{L}\,dx = \frac{\varepsilon}{L}\left[\left[-\frac{L}{\pi(m+\tfrac{1}{2})}\cos\left[(m+\tfrac{1}{2})\frac{\pi x}{L}\right]\cdot x\right]_0^L + \frac{L}{(m+\tfrac{1}{2})\pi}\int_0^L \cos\left[(m+\tfrac{1}{2})\frac{\pi x}{L}\right]\cdot dx\right]$$

$$= \frac{\varepsilon}{L}\left[-\frac{L}{\pi(m+\tfrac{1}{2})}\underbrace{\cos\left[(m+\tfrac{1}{2})\pi\right]}_{=0}\cdot L + \frac{L^2}{(m+\tfrac{1}{2})^2\pi^2}\left[\sin(m+\tfrac{1}{2})\frac{\pi x}{L}\right]_0^L\right] = \frac{\varepsilon L}{(m+\tfrac{1}{2})^2\pi^2}\underbrace{\sin(m+\tfrac{1}{2})\pi}_{=(-1)^m}$$



$$\Rightarrow A_m = \frac{2\varepsilon}{(m+\tfrac{1}{2})^2\pi^2}(-1)^m \quad \& \quad z(x,t) = \frac{2\varepsilon}{\pi^2}\sum_{n=0}^{\infty}(-1)^n\frac{\sin(n+\tfrac{1}{2})\frac{\pi x}{L}\cos(n+\tfrac{1}{2})\frac{\pi ct}{L}}{(n+\tfrac{1}{2})^2}$$

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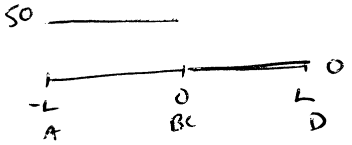
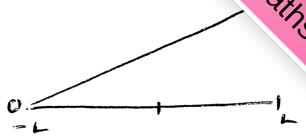
9) $z_{xx} = \dfrac{1}{c^2}(z_{tt} + \lambda z_t)$, $z = X(x)T(t)$, $\Rightarrow \dfrac{X''}{X} = \dfrac{1}{c^2}\left(\dfrac{T''}{T} + \dfrac{\lambda T'}{T}\right)$

BC $\Rightarrow X(0) = X(L) = 0$ so $X(x) = A\sin\dfrac{n\pi x}{L}$ with $\dfrac{X''}{X} = \dfrac{n^2\pi^2}{L^2}(-1)$. So $T(t)$ satisfies $T'' + \lambda T' + \dfrac{c^2 n^2\pi^2}{L^2}T = 0$. Look for solutions $e^{\alpha t}$ & $\alpha^2 + \lambda\alpha + \dfrac{c^2n^2\pi^2}{L^2} = 0$

$\Rightarrow \alpha = -\dfrac{\lambda}{2} \pm \dfrac{1}{2}\sqrt{\lambda^2 - \dfrac{4c^2n^2\pi^2}{L^2}}$ & if λ is small enough for roots to be complex, they are $\alpha = -\dfrac{\lambda}{2} \pm iq_n$ with $q_n^2 = \dfrac{1}{4}\cdot\dfrac{4c^2n^2\pi^2}{L^2} - \lambda^2/4$. as required & $T(t) = A_n\cos(q_n t + \varepsilon_n)e^{-\lambda t/2}$

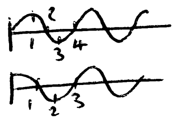
Sheet 1

3) 

Initial conditions are as shown here. Boundary conditions suggest an ultimate steady solution shown → 

ie $\theta = \theta_S(x) = 50 + 50x/L$. Steady solutions can only be linear functions of x.

If we write $\theta = \theta_S + \theta_u(x,t)$ then $\theta_{ut} = \alpha\,\theta_{uxx}$ and initially $\theta_u = \begin{matrix} 50 & x<0 \\ 0 & x>0 \end{matrix} - \theta_S$ ie $\theta_u = -50x/L$ for $x<0$ & $-50 - 50x/L$ for $x>0$, with boundary conditions $\theta_u = 0$ at $x=-L$ & $x=L$. Looking for a solution $\theta_u = X(x)T(t)$ yeilds $XT' = \alpha X''T$, $T'/\alpha T = X''/X = -\lambda^2$, choosing negative separation constants. Therefore $T = Ae^{-\alpha\lambda^2 t}$ and $X'' + \lambda^2 X = 0$, $X = A\cos x\lambda + B\sin x\lambda$. The boundary conditions at $x=-L$ & $x=L$ yeild $A\cos\lambda L + B\sin\lambda L = 0$, $A\cos\lambda L - B\sin\lambda L = 0$, $\begin{pmatrix} \cos\lambda L & \sin\lambda L \\ \cos\lambda L & -\sin\lambda L \end{pmatrix}\begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$
$\Rightarrow -2\sin\lambda L\cos\lambda L = 0 \Rightarrow \sin 2\lambda L = 0$, $2\lambda L = n\pi$, $\lambda = n\pi/2L$, $n = 1,2,\dots$ with these choices of λ we get $\begin{pmatrix} \cos n\pi/2 & \sin n\pi/2 \\ \cos n\pi/2 & -\sin n\pi/2 \end{pmatrix}\begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ & if n is odd, $n = 2m+1$ say, $m = 0,1,2\dots$ this is $\begin{pmatrix} 0 & (-1)^m \\ 0 & -(-1)^m \end{pmatrix}\begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow B = 0$ & $X(x) = A\cos\frac{(2m+1)\pi x}{2L}$



If n is even, $n = 2m$ we have $\begin{pmatrix} (-1)^m & 0 \\ (-1)^m & 0 \end{pmatrix}\begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, $A = 0$ & $X(x) = B\sin\frac{m\pi x}{L}$

The general solution is therefore $\theta_u = \sum_{m=1}^{\infty} A_m\cos\frac{(2m+1)\pi x}{2L}e^{-\alpha(2m+1)^2\pi^2 t/4L^2} + \sum_{m=1}^{\infty} B_m\sin\frac{m\pi x}{L}e^{-\alpha m^2\pi^2 t/L^2}$

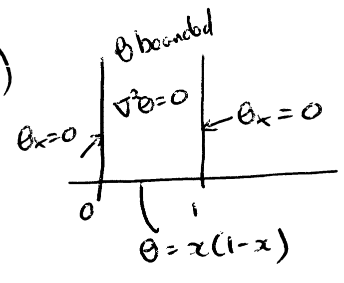
Now we want only the solution at $x=0$, $\theta = \theta_S + \theta_u = 50 + \sum_{m=1}^{\infty} A_m e^{-\alpha(2m+1)^2\pi^2 t/4L^2}$ & need to calculate A_m only. At $t=0$ we have

$$\theta_u(x,0) = \begin{matrix} -50x/L & x<0 \\ -50 - 50x/L & x>0 \end{matrix} = \sum_1^{\infty} A_m\cos\frac{(2m+1)\pi x}{2L} + \sum_1^{\infty} B_m\sin\frac{m\pi x}{L},$$

describing the even & odd parts of the initial condition respectively. ~~The even part of the initial condition is~~ We are not interested in B_m but multiplying by $\cos\frac{(2n+1)\pi x}{2L}$ & integrating gives $\sum_1^{\infty} A_m\int_{-L}^{L}\cos\frac{(2m+1)\pi x}{2L}\cos\frac{(2n+1)\pi x}{2L}\,dx = A_n . 2L . \frac{1}{2} = \int_{-L}^{L}\left\{\begin{matrix} -50x/L & x<0 \\ -50 - 50x/L & x>0 \end{matrix}\right\}\cos\frac{(2n+1)\pi x}{2L}$

$$\Rightarrow A_n = \frac{1}{L}\left[\int_{-L}^{L}\underbrace{-\frac{50x}{L}}_{\text{odd}}\underbrace{\cos\frac{(2n+1)\pi x}{2L}}_{\text{even}}dx + \int_0^L -50\cos\frac{(2n+1)\pi x}{2L}\right] = -\frac{50}{L}\cdot\frac{2L}{(2n+1)\pi}\left[\sin\frac{(2n+1)\pi x}{2L}\right]_0^L$$

$$= -\frac{100}{\pi}\frac{1}{2n+1}\sin(2n+1)\frac{\pi}{2} = -\frac{100}{\pi}\frac{(-1)^n}{2n+1} \text{ as required.}$$

4) 

Looking for a solution $\theta(x,y) = X(x)Y(y)$ gives $\frac{X''}{X} = -\frac{Y''}{Y} = \text{const}$
$X'(0) = X'(1) = 0$, Y bounded as $y \to \infty$.
We choose the separation constant to be $-$ve, $= -\lambda^2$ say, so that we can impose $X'(0) = X'(1) = 0$ & require $X'' + \lambda^2 X = 0$, $Y'' - \lambda^2 Y = 0$ & $X = A\cos\lambda x + B\sin\lambda x$.
$X'(0) = 0 \Rightarrow B = 0$ & $X'(1) = 0 \Rightarrow -A\lambda\sin\lambda = 0 \Rightarrow \lambda = n\pi$.
$Y'' - \lambda^2 Y = 0 \Rightarrow Y = Ae^{-\lambda y} + Be^{\lambda y}$ & for Y bounded we need $B = 0$.

We need, in this example, to include the zero separation constant, leading $x''=0 \quad x'(0)=x'(1)=0$ ie $x = \text{Constant} = A_0$ say. This gives

$$\theta = A_0 + \sum_{n=1}^{\infty} A_n \cos n\pi x\, e^{-n\pi y}$$

with $A_0, A_1, \dots$ chosen so that $\theta(x,0) = A_0 + \sum_1^{\infty} A_n \cos n\pi x = x(1-x)$.

Integrating gives $\int_0^1 A_0\,dx + \sum_1^{\infty} A_n \underbrace{\int_0^1 \cos n\pi x\,dx}_{0} = \int_0^1 x(1-x)\,dx = \frac{1}{2} - \frac{1}{3} = \frac{1}{6}$

$\Rightarrow A_0 = 1/6$

& multiplying by $\cos m\pi x$ & integrating: $A_m \cdot \frac{1}{2} = \int_0^1 \cos m\pi x \cdot x(1-x)\,dx$

$$= \Big[\underbrace{\frac{\sin m\pi x}{m\pi}(x)(1-x)}_{(=0)}\Big]_0^1 - \int_0^1 \frac{\sin m\pi x}{m\pi}(1-2x)\,dx = \Big[\frac{\cos m\pi x}{m^2\pi^2}(1-2x)\Big]_0^1 + \frac{2}{m^2\pi^2}\underbrace{\int_0^1 \cos m\pi x\,dx}_{0}$$

$$= \frac{1}{m^2\pi^2}\{\cos m\pi(-1) - \cos 0\} = -\frac{1}{m^2\pi^2}\left((-1)^m + 1\right) \;\&\; \theta(x,y) = \frac{1}{6} - \sum_{n=1}^{\infty} \frac{\cos n\pi x\, e^{-n\pi y}}{n^2\pi^2}\cdot[(-1)^n+1]\cdot 2$$

The solution is required at $x = 1/2$ & $\cos n\pi/2 = 0$ if n is odd, $(-1)^{n/2}$ if n is even. So we only need the terms for even n & writing $n = 2m$ we get, with $n/2 = m$,
$(-1)^n + 1 = (-1)^{2m} + 1 = 1 + 1 = 2$, $n^2\pi^2 = 4m^2\pi^2$, $\cos 2m\pi \cdot \frac{1}{2} = \cos m\pi = (-1)^m$

$$\theta(1/2, y) = \frac{1}{6} - \sum_{m=1}^{\infty} \frac{(-1)^m e^{-2m\pi y}}{m^2\pi^2}.$$